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Operators on Pavelka's algebras induced by fuzzy relations

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Abstract

The theory of operators generated by binary fuzzy relations is quickly growing due its fundamental nature and potential applicability. The main goal of the paper is to present several representation theorems for operators induced by fuzzy relations on Pavelka's algebras (for example closure operators used in formal concept analysis, monadic operators or tense operators). Consequently we establish algebraic models with their semantics which are usable in the non-classical logic research and in the computer science research.

Keywords: Galois connection; Fuzzy relation; Tense operators; Monadic operators; Residuated lattice; Pavelka's algebra

1. Introduction

The fuzzy logic brings theoretic background to analyze non-boolean inputs. The classical mathematical logic loses its applicability in the case when there is no natural way to describe the analyzed data, properties or relations using 0/1 (false/true). A key idea is to extend the two element truth scale to a richer one. Logics obtained by the extensions are called fuzzy logics. There exist several theoretic concepts introducing a logic with vagueness or uncertainty. In this paper we use the concept of a *commutative bounded integral residuated lattice*

$$\mathbf{A} = (A; \vee, \wedge, \cdot, \rightarrow, 0, 1)$$

which we will call simply a *residuated lattice*. Thus,

i) $(A; \vee, \wedge, 0, 1)$ is a bounded lattice,
ii) $(A; \cdot, 1)$ is a commutative monoid,
iii) the adjointness property holds, i.e.,

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$$x \cdot y \leq z \text{ if, and only if, } x \leq y \to z.$$

We understand this structure as a general algebraic model of a truth scale. The elements 0 and 1 model the strict false and the strict true. Other elements represent the fuzzy truth degrees. The connective of logical conjunction is modelled by the operation $\cdot$, the implication by $\to$.

Important models are residuated lattices induced by triangular norms. By a *triangular norm* we mean a binary operation $\cdot$ defined on the real interval [0, 1] which is commutative, associative and left-continuous (in usual sense) and monotone. Then the residual operation is given by

$$x \to y = \max\{a \mid a \cdot x \leq y\}.$$

Used order is the standard one.

Example 1. The Łukasiewicz triangular norm is defined by $x \cdot y = \min\{1 - x - y, 0\}$. Its residual operation is $x \to y = \min\{1 - x + y, 1\}$.

Example 2. The Gödel triangular norm is defined by $x \cdot y = \min\{x, y\}$ and its residual operation is $x \to y = y$ if $y \leq x$ and $x \to y = 1$ if $x < y$.

Example 3. Product triangular norm is just the standard product of real numbers and its residual operation is $x \to y = \min\{x/y, 1\}$ if $y \neq 0$ and $x \to 0 = 0$ if $x \neq 0$ and $0 \to 0 = 1$.

It is well-known that all continuous triangular norms are decomposable into Łukasiewicz, Gödel and product ones (for the details we refer to [25]). Moreover, the Hájek's basic logic is generated exactly by the residuated lattices induced by the continuous triangular norms [14]. Altogether, triangular norms, the Hájek's basic logic, or its special subclasses are the most applicable classes of fuzzy logics in the computer science.

Algebraic models of the Hájek's basic logic are the, so called, *BL-algebras* which are just residuated lattices satisfying the divisibility law

$$x \cdot (x \to y) = x \wedge y$$

and the prelinearity law

$$(x \to y) \vee (y \to x) = 1.$$

BL-algebras satisfying the double negation law $\neg\neg x = x$ where

$$\neg x := x \to 0$$

are called MV-algebras. The class of MV-algebras is induced exactly by the Łukasiewicz triangular norm.

Recall that MV-algebras are usually defined as algebras of type $\mathbf{A} = (A; \oplus, \neg, 0)$ such that

(MV1) $(A; \oplus, 0)$ is a commutative monoid,
(MV2) the double negation $\neg\neg x = x$ holds,
(MV3) the Łukasiewicz axiom $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ holds.

However, both presented definitions are equivalent. More precisely, if $\mathbf{A} = (A; \vee, \wedge, \cdot, \to, 0, 1)$ is a residuated lattice satisfying divisibility, prelinearity, and double negation law. Then an algebra $(A; \oplus, \neg, 0)$ where $x \oplus y := \neg(\neg x \cdot \neg y)$ is an MV-algebra. Conversely, let us have an MV-algebra $\mathbf{A} = (A; \oplus, \neg, 0)$ then an algebra $(A; \vee, \wedge, \cdot, \to, 0, 1)$, where

$$x \vee y := \neg(\neg x \oplus y) \oplus y,$$
$$x \wedge y := \neg(\neg x \vee \neg y),$$
$$x \cdot y := \neg(\neg x \oplus \neg y),$$
$$x \to y = \neg x \oplus y,$$
$$1 := \neg 0$$

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