ARTICLE IN PRESS
FSS:6951

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems ••• (••••) •••–•••

FUZZY sets and systems

www.elsevier.com/locate/fss

Variety theorem for algebras with fuzzy orders

Vilem Vychodil *

Dept. Computer Science, Palacky University Olomouc, Czech Republic

Received 3 September 2014; received in revised form 19 November 2015; accepted 21 November 2015

Abstract

We present generalization of the Bloom variety theorem of ordered algebras in fuzzy setting. Algebras with fuzzy orders consist of sets of functions which are compatible with fuzzy orders. Fuzzy orders are defined on universe sets of algebras using complete residuated lattices as structures of degrees. In this setting, we show that classes of models of fuzzy sets of inequalities are closed under suitably defined formations of subalgebras, homomorphic images, and direct products. Conversely, we prove that classes having these closure properties are definable by fuzzy sets of inequalities.

Keywords: General algebra; Variety theorem; Complete residuated lattice; Fuzzy order

1. Introduction

In this paper, we develop previous results on closure properties of model classes of fuzzy structures [3,7] which focused on algebras equipped with fuzzy equalities. The notion of an algebra with fuzzy equality appeared in [1] and has been further investigated in [2,4,38]. In our paper, we refer to [6] which contains detailed description of the structures and their algebraic properties. Recall from [6] that algebras with fuzzy equalities are considered as general algebras, i.e., structures consisting of universe sets equipped with (ordinary) n-ary functions, which are in addition compatible with given fuzzy equality relations. Thus, an algebra with fuzzy equality may be viewed as a traditional algebra with an additional relational component. The aim of algebras with fuzzy equalities is to formalize functional systems which preserve similarity in that the functions, when used with pairwise similar arguments, produce similar results. The existing results on closure properties of algebras with fuzzy equalities include generalization [3] of the Birkhoff variety theorem [10] and results on classes definable by graded implications between identities [7], cf. also [5] for a survey of results.

Our aim in this paper is to study algebras equipped with fuzzy orders [39] which in general are intended to formalize different relationships than similarity fuzzy relations and may be regarded as representations of (graded) preferences. Analogously as in the case of algebras with fuzzy equalities, we assume that functions of algebras are compatible

* Tel.: +420 585 634 705; fax: +420 585 411 643.
E-mail address: vychodil@binghamton.edu.

http://dx.doi.org/10.1016/j.fss.2015.11.017

Please cite this article in press as: V. Vychodil, Variety theorem for algebras with fuzzy orders, Fuzzy Sets and Systems (2016), http://dx.doi.org/10.1016/j.fss.2015.11.017

with given fuzzy orders. In this case, the compatibility says that the degree to which the result of a function applied to $b_1, \ldots, b_n$ is preferred to the result of the function applied to $a_1, \ldots, a_n$ is at least as high as the degree to which b_1 is preferred to a_1 *and* $\cdots$ *and* b_n is preferred to a_n with the logical connective "*and*" interpreted by a suitable truth function, e.g., a left-continuous triangular norm [29].

One of the important aspects of our approach is that we consider theories as fuzzy sets of atomic formulas (inequalities) prescribing degrees to which the formulas shall be satisfied in models. We thus utilize one of the basic concepts of Pavelka's abstract fuzzy logic [33–35] in which theories are fuzzy sets of (abstract formulas) and the semantic entailment is defined to degrees, cf. also [23] for a general treatment of related topics. In this setting, we prove that model classes of fuzzy sets of inequalities are closed under formations of homomorphic images, subalgebras, and direct products and, conversely, classes obeying these closure properties are model classes of fuzzy sets of inequalities, establishing an analogy of the Bloom variety theorem [11] for ordered algebras. Let us note that our approach is conceptually different from the approaches to fuzzy algebras which started with [36] and were further generalized, e.g., in [32], and consider fuzzy subsets on universe sets of structures. The conceptual differences are described in [6], see also [5, Section 2.9] for a detailed comparison.

This paper is organized as follows. In Section 2, we present preliminaries. In Section 3, we recall the algebras with fuzzy orders and present details on their relationship to ordinary algebras and algebras with fuzzy equalities. Section 4 presents examples of algebras with fuzzy orders. Section 5 summarizes basic algebraic constructions which are further exploited in Section 6 devoted to the relationship of varieties and inequational classes of algebras with fuzzy orders. Conclusion and outline of future work is presented in Section 7.

2. Preliminaries

A complete (integral commutative) residuated lattice [2,22] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ where $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes$ and $\rightarrow$ satisfy the adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ ($a, b, c \in L$). Note that complete residuated lattices are also known as integral, commutative cl-monoids and commutative, strictly two-sided quantales, see [18] and the references therein. Examples of complete residuated lattices include structures on the real unit interval given by left-continuous t-norms [29] as well as finite structures and structures playing important role in fuzzy logics in the narrow sense [20,25,26]. A survey of recent results in fuzzy logics can be found in [16,17]. Further in the paper, $\mathbf{L}$ always refers to a complete residuated lattice.

Given $M \neq \emptyset$, a binary $\mathbf{L}$-relation R on M is a map $R\colon M \times M \rightarrow L$. For $a, b \in M$, the degree $R(a, b) \in L$ is interpreted as the degree to which a and b are R-related. If $\approx, \preccurlyeq, \ldots$ denote binary $\mathbf{L}$-relations, we use the usual infix notation and write $a \approx b$ instead of $\approx(a, b)$ and the like. For binary $\mathbf{L}$-relations R_1 and R_2 on M, we put $R_1 \subseteq R_2$ whenever $R_1(a, b) \leq R_2(a, b)$ for all $a, b \in M$ and say that R_1 is (fully) contained in R_2. As usual, a binary $\mathbf{L}$-relation R^{-1} satisfying $R^{-1}(a, b) = R(b, a)$ (for all $a, b \in M$) is called the inverse of R. For convenience, inverses of relations denoted by symbols like $\preccurlyeq, \ldots$ are denoted by $\succcurlyeq, \ldots$. Operations with $\mathbf{L}$-relations can be defined componentwise [2, p. 80] using operations in $\mathbf{L}$. For instance, for an I-indexed family $\{R_i;\ i \in I\}$ of binary $\mathbf{L}$-relations on M, we may consider the intersection $\bigcap_{i\in I} R_i$ of all R_i ($i \in I$) defined by $\bigl(\bigcap_{i\in I} R_i\bigr)(a, b) = \bigwedge_{i\in I} R_i(a, b)$ for all $a, b \in M$. A binary $\mathbf{L}$-relation R_1 on M is called the symmetric interior of a binary $\mathbf{L}$-relation R_2 on M whenever $R_1 = R_2 \cap R_2^{-1}$.

In the paper, we consider algebras of a given type. A type is given by a set F of function symbols together with their arities. We assume that arity of each $f \in F$ is finite. Recall that an algebra (of type F) is a structure $\mathbf{M} = \langle M, F^{\mathbf{M}}\rangle$ where M is a non-empty universe set and $F^{\mathbf{M}}$ is a set of functions interpreting the function symbols in F. Namely,

$$F^{\mathbf{M}} = \bigl\{f^{\mathbf{M}}\colon M^n \rightarrow M;\ f \text{ is } n\text{-ary function symbol in } F\bigr\}.$$

An algebra with $\mathbf{L}$-equality [6, Definition 3.1] (of type F) is a structure $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ such that $\langle M, F^{\mathbf{M}}\rangle$ is an algebra (of type F) and $\approx^{\mathbf{M}}$ is a binary $\mathbf{L}$-relation on M satisfying the following conditions:

$$a \approx^{\mathbf{M}} b = 1 \text{ iff } a = b, \tag{1}$$

$$a \approx^{\mathbf{M}} b = b \approx^{\mathbf{M}} a, \tag{2}$$

$$a \approx^{\mathbf{M}} b \otimes b \approx^{\mathbf{M}} c \leq a \approx^{\mathbf{M}} c, \tag{3}$$

$$a_1 \approx^{\mathbf{M}} b_1 \otimes \cdots \otimes a_n \approx^{\mathbf{M}} b_n \leq f^{\mathbf{M}}(a_1, \ldots, a_n) \approx^{\mathbf{M}} f^{\mathbf{M}}(b_1, \ldots, b_n) \tag{4}$$

Please cite this article in press as: V. Vychodil, Variety theorem for algebras with fuzzy orders, Fuzzy Sets and Systems (2016), http://dx.doi.org/10.1016/j.fss.2015.11.017