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Short Communication

General coupled semirings of residuated lattices ☆

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Received 7 March 2014; received in revised form 19 October 2015; accepted 14 December 2015

Abstract

Di Nola and Gerla showed that MV-algebras and coupled semirings are in a natural one-to-one correspondence. We generalize this correspondence to residuated lattices satisfying the double negation law.

Keywords: Residuated lattice; Double negation law; Semiring; General coupled semiring

It was shown by Di Nola and Gerla [6,7] that to every MV-algebra there can be assigned a so-called coupled semiring which bears all the information on that MV-algebra, i.e., the latter can be recovered by its assigned coupled semiring. This fact inspired us to modify the concept of a coupled semiring in order to get a similar representation for commutative basic algebras [4] or for general basic algebras [5].

Every MV-algebra is indeed a residuated lattice satisfying the double negation law, the prelinearity and the divisibility condition (see [2] for details). Hence we try to find a representation by means of some sort of coupled semirings also for the more general class of residuated lattices. In fact, we are successful in the case where the double negation law is assumed.

This shows that the construction of a coupled semiring from [6] and [7] is quite general and it can be applied in the fairly general case of residuated lattices satisfying the double negation law. For similar categorical considerations see [1].

Finally, we want to stress the importance of semirings treated in the paper in applications and in the context of tropical geometry, see e.g. [10].

We start with the definition of a residuated lattice.

☆ Support of the research of both authors by the bilateral project "New perspectives on residuated posets", supported by the Austrian Science Fund (FWF), project I 1923-N25, and the Czech Science Foundation (GAČR), project 15-34697L, as well as by the project "Ordered structures for algebraic logic", supported by AKTION Austria–Czech Republic, project 71p3, is gratefully acknowledged.
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http://dx.doi.org/10.1016/j.fss.2015.12.009
0165-0114/© 2015 Published by Elsevier B.V.

Definition 1. A *residuated lattice* is an algebra $\mathcal{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following axioms for all $x, y, z \in L$:

(i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
(ii) $(L, \otimes, 1)$ is a commutative monoid.
(iii) $x \leq y \rightarrow z$ if and only if $x \otimes y \leq z$

Remark 2. Condition (iii) is called the *adjointness property*.

As a source for elementary properties of residuated lattices see the monograph by Bělohlávek [2]. We will work with residuated lattices having one more property.

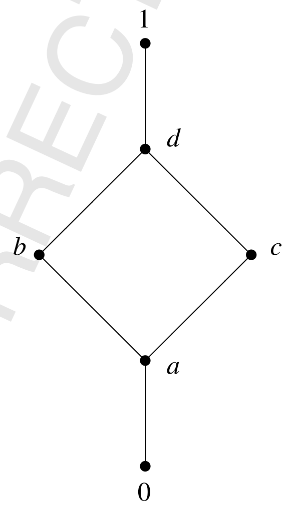
Definition 3. Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. On L we define two further operations as follows:

$$\neg x := x \rightarrow 0 \text{ and}$$
$$x \oplus y := \neg(\neg x \otimes \neg y)$$

for all $x, y \in L$. Further, we say that $\mathcal{L}$ satisfies the *double negation law* if $\neg\neg x = x$ for all $x \in L$.

If $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a residuated lattice satisfying the double negation law then $(L, \oplus, \neg, 0)$ need not be an MV-algebra. This can be seen from the following example:

Example 4. (Cf. [11].) If $(L, \vee, \wedge, 0, 1)$ denotes the bounded lattice given by the following Hasse diagram:



and we define binary operations $\otimes$ and $\rightarrow$ on L as follows:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	b	0	b	b
c	0	0	0	c	c	c
d	0	0	b	c	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	c	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

then we have

Please cite this article in press as: I. Chajda, H. Länger, General coupled semirings of residuated lattices, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2015.12.009