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Single chain completeness and some related properties

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Abstract

In 2010 Franco Montagna investigated two interesting properties of the axiomatic extensions of MTL, the single chain completeness (SCC) and the strong single chain completeness (SSCC). An axiomatic extension L of MTL enjoys the SCC if there is an L-chain $\mathcal{A}$ s.t. L is complete w.r.t. $\mathcal{A}$, and L enjoys the SSCC if there is an L-chain $\mathcal{A}$ s.t. L is strongly complete w.r.t. $\mathcal{A}$. Clearly the SSCC implies the SCC, whilst the converse implication has been left as an open problem. In this work we show that the SCC does not imply the SSCC, and that the SCC and SSCC are strongly related to some logical and algebraic properties relevant for substructural logics, as Halldén completeness (HC) and Deductive Maksimova variable separation property (DMVP). The HC will provide a logical characterization for the SCC, for every axiomatic extension of MTL, whilst the DMVP will be proved to be equivalent to the SSCC, for the n-contractive axiomatic extensions of BL. We conclude by studying the axiomatic extensions of MTL expanded with the Δ operator, by showing that SCC and SSCC always coincide, even in the first-order case.
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Keywords: Many-valued logics; Monoidal t-norm based logic; Single chain completeness; Variable separation properties; Substructural logics; Halldén completeness

1. Introduction

Monoidal t-norm based logic (MTL) was introduced in [12] as the basis of a framework of many-valued logics initially studied by Petr Hájek in [16].

The initial aim was to study a framework of logics induced by continuous t-norms and their residua (BL), and by left-continuous t-norm and their residua (MTL): however, the algebraic semantics of each of these logics is far richer. In particular, MTL and its axiomatic extensions, logics obtained from it by adding other axioms, are all algebraizable in the sense of [6] and their corresponding classes of algebras form an algebraic variety (see [23,9]). Given an axiomatic extension L of MTL one can study the completeness properties of L with respect to some classes of L-algebras: for example the class of all L-algebras, of all L-chains, or the class of t-norm based L-algebras. As shown in [16,12] every extension L of MTL is strongly complete w.r.t. the class of all L-chains. Why is it important to find completeness

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results, for a logic, with respect to a class of totally ordered algebras? As pointed out by Petr Hájek in the introduction of his book [16] a desirable characteristic in a framework of many-valued logics is the comparative notion of truth: that is, sentences may be compared according to their truth values. So, if we agree with this point of view, then we must necessarily deal with totally ordered algebras, since the "truth values" must be comparable. However, the class of all L-chains may be very large and we could not have a single algebraic structure where to evaluate the truth values of formulas.

This problem can be overcome when an extension L of MTL is complete w.r.t. an L-chain: in this case we say that L enjoys the single chain completeness (SCC). We say that L enjoys the strong single chain completeness (SSCC) if L is strongly complete w.r.t. some L-chain. The article [21] presents a systematic study of completeness properties of this type, for the axiomatic extensions of MTL, but some problems remain open. In particular, it is clear that the SSCC implies the SCC, but the converse implication is left in [21] as an open problem.

In this paper we study the relations between the SCC, SSCC and some logical and algebraic properties presented in [14] for substructural logics, solving the open problem left by Franco Montagna and obtaining other results. Some of the properties studied in [14] were firstly introduced in the area of modal and superintuitionistic logics, as the Halldén completeness (HC), the Maksimova variable separation property (MVP), and deductive Maksimova variable separation property (DMVP). After some preliminary results in Sections 2–4, in Section 5 we will show that the SCC and HC are equivalent, that the SSCC implies the DMVP, whilst the DMVP and SSCC are equivalent for the n-contractive axiomatic extensions of BL. This means that HC and DMVP provide a logical characterization for the SCC and SSCC, for some families of logics. We will also show that the MVP fails in every axiomatic extension of MTL not extending SMTL: via this result we will be able to solve an open problem stated in [14]. In Section 6 we will give a negative answer to the open problem left by Franco Montagna, by showing an axiomatic extension of WNM – whose variety is generated by a chain of five elements – in which the SCC holds, whilst the DMVP and the SSCC fail. We will also show that such counterexample is minimal. Section 7 will be devoted to a particular type of single chain completeness, and we will also solve an open problem mentioned in [19]. We will conclude with Section 8, by analyzing the expansions of MTL with the Δ operator, a topic left in [21] as future work. We will show that for such logics SCC and SSCC are equivalent, even in the first-order case.

2. Preliminaries

2.1. Syntax

Monoidal t-norm based logic (MTL) was introduced in [12]: it is based over connectives $\{\&, \wedge, \to, \bot\}$ (the first three are binary, whilst the last one is 0-ary), and a denumerable set of propositional variables. Useful derived connectives are the negation $\neg\varphi \overset{\text{def}}{=} \varphi \to \bot$, the top $\top \overset{\text{def}}{=} \neg\bot$, and the disjunction $\varphi \vee \psi \overset{\text{def}}{=} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$. The notion of formula is defined inductively in the usual way.

MTL can be axiomatized with the following axioms.

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \tag{A1}$$

$$(\varphi \& \psi) \to \varphi \tag{A2}$$

$$(\varphi \& \psi) \to (\psi \& \varphi) \tag{A3}$$

$$(\varphi \wedge \psi) \to \varphi \tag{A4}$$

$$(\varphi \wedge \psi) \to (\psi \wedge \varphi) \tag{A5}$$

$$(\varphi \& (\varphi \to \psi)) \to (\psi \wedge \varphi) \tag{A6}$$

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi) \tag{A7a}$$

$$((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi)) \tag{A7b}$$

$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \tag{A8}$$

$$\bot \to \varphi \tag{A9}$$

As inference rule we have modus ponens: $\frac{\varphi \quad \varphi \to \psi}{\psi}$.

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