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Lattice operations on fuzzy implications and the preservation of the exchange principle

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Received 8 October 2015; received in revised form 22 February 2016; accepted 25 February 2016

Abstract

In this work, we solve an open problem related to the preservation of the exchange principle (EP) of fuzzy implications under lattice operations ([3], Problem 3.1.). We show that generalizations of the commutativity of antecedents (CA) to a pair of fuzzy implications (I, J), viz., the generalized exchange principle and the mutual exchangeability are sufficient conditions for the solution of the problem. Further, we determine conditions under which these become necessary too. Finally, we investigate the pairs of fuzzy implications from different families such that (EP) is preserved by the join and meet operations.

Keywords: The commutativity of antecedents; Fuzzy implication; The exchange principle; The generalized exchange principle; The mutual exchangeability; Lattice operations

1. Introduction

Fuzzy implications are one of the important logical connectives in fuzzy logic. These operators generalize the classical implication from $\{0, 1\}$-setting to many valued setting. Fuzzy implications on the unit interval $[0, 1]$ are defined as follows:

Definition 1.1. (See [1], Definition 1.1.1.) A function $I : [0, 1]^2 \longrightarrow [0, 1]$ is called a *fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, the following conditions:

$$\text{if } x_1 \le x_2, \text{ then } I(x_1, y) \ge I(x_2, y)\,, \tag{I1}$$

$$\text{if } y_1 \le y_2, \text{ then } I(x, y_1) \le I(x, y_2)\,, \tag{I2}$$

$$I(0, 0) = 1\,,\ I(1, 1) = 1\,,\ I(1, 0) = 0\,. \tag{I3}$$

Let $\mathbb{I}$ denote the set of fuzzy implications defined on $[0, 1]$.

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http://dx.doi.org/10.1016/j.fss.2016.02.017
0165-0114/

Please cite this article in press as: N.R.Vemuri, B. Jayaram, Lattice operations on fuzzy implications and the preservation of the exchange principle, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.02.017

Table 1
Examples of fuzzy implications (cf. Table 1.3 in [1]).

Name	Formula	(EP)
Łukasiewicz	$I_{\mathbf{LK}}(x,y)=\min(1,1-x+y)$	✓
Gödel	$I_{\mathbf{GD}}(x,y)=\begin{cases}1, & \text{if } x\le y\\ y, & \text{if } x>y\end{cases}$	✓
Reichenbach	$I_{\mathbf{RC}}(x,y)=1-x+xy$	✓
Kleene–Dienes	$I_{\mathbf{KD}}(x,y)=\max(1-x,y)$	✓
Goguen	$I_{\mathbf{GG}}(x,y)=\begin{cases}1, & \text{if } x\le y\\ \frac{y}{x}, & \text{if } x>y\end{cases}$	✓
Rescher	$I_{\mathbf{RS}}(x,y)=\begin{cases}1, & \text{if } x\le y\\ 0, & \text{if } x>y\end{cases}$	×
Fodor	$I_{\mathbf{FD}}(x,y)=\begin{cases}1, & \text{if } x\le y\\ \max(1-x,y), & \text{if } x>y\end{cases}$	✓
Least FI	$I_{\mathbf{0}}(x,y)=\begin{cases}1, & \text{if } x=0 \text{ or } y=1\\ 0, & \text{if } x>0 \text{ and } y<1\end{cases}$	×
Greatest FI	$I_{\mathbf{1}}(x,y)=\begin{cases}1, & \text{if } x<1 \text{ or } y>0\\ 0, & \text{if } x=1 \text{ and } y=0\end{cases}$	✓

1.1. Fuzzy implications and the exchange principle (EP)

Let $\longrightarrow$ denote the classical implication. Then, from classical logic, it follows that

$$p \longrightarrow (q \longrightarrow r) \equiv q \longrightarrow (p \longrightarrow r), \tag{CA}$$

which is known as the commutativity of antecedents (CA).

Note that, a straightforward generalization of (CA) to the many-valued setting need not hold true always and thus leads to the notion of *the exchange principle* (EP) of a fuzzy implication, which is defined as follows:

Definition 1.2. (See cf. [1], Definition 1.3.1.) A fuzzy implication I is said to satisfy *the exchange principle* (EP), if for all $x, y, z \in [0,1]$,

$$I(x, I(y,z)) = I(y, I(x,z)) . \tag{EP}$$

Let $\mathbb{I}_{\mathbf{EP}}$ denote the set of fuzzy implications satisfying (EP).

Table 1 (see also, Table 1.3 in [1]) lists some examples of basic fuzzy implications along with whether they satisfy the exchange principle (EP) or not.

1.2. Motivation for this work

In this work, we investigate the problem (Problem 3.1 in [3], which is stated below as Problem 1.4) of preservation of (EP) by the lattice operations of fuzzy implications, which are defined as in the following result:

Theorem 1.3. *(See [1], Theorem 6.1.1.) The family* $(\mathbb{I}, \le)$ *is a complete, completely distributive lattice with the lattice operations*

$$(I \vee J)(x,y) := \max(I(x,y), J(x,y)) , \qquad x, y \in [0,1] ,$$
$$(I \wedge J)(x,y) := \min(I(x,y), J(x,y)) , \qquad x, y \in [0,1] ,$$

where $I, J \in \mathbb{I}$.