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A unified approach to the monotone convergence theorem for nonlinear integrals ☆

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Abstract

We present a unified approach to the monotone convergence theorem for nonlinear integrals such as the Choquet, the Šipoš, the Sugeno, and the Shilkret integral. A nonlinear integral may be viewed as a nonlinear functional defined on a set of pairs of a nonadditive measure and a measurable function. We thus formulate our general type of monotone convergence theorem for such a functional. The key tool is a perturbation of functional that manages not only the monotonicity of the functional but also the small change of the functional value arising as a result of adding small amounts to a measure and a function in the domain of the functional. Our approach is also applicable to the Lebesgue integral when a nonadditive measure is σ-additive.

Keywords: Nonadditive measure; Nonlinear integral; Monotone convergence theorem; Integral functional; Perturbation

1. Introduction

The monotone convergence theorem for the abstract Lebesgue integral states that if $\{f_n\}_{n\in\mathbb{N}}$ is an increasing sequence of measurable, extended real-valued non-negative functions on a measure space $(X, \mathcal{A}, \mu)$ with pointwise limit f, then $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$, and is referred to as the monotone increasing convergence theorem (for short, MICT). Since the Lebesgue integral is linear, namely $\int_X (f_1 - f_n) d\mu = \int_X f_1 d\mu - \int_X f_n d\mu$ $(n = 1, 2, \ldots)$, we also obtain the same conclusion for a decreasing $\{f_n\}_{n\in\mathbb{N}}$ with $\int_X f_1 d\mu < \infty$, which is in turn referred to as the monotone decreasing convergence theorem (for short, MDCT). The monotone convergence theorem has played a fundamental role in the study of limit theorems of integral and in fact yields other basic theorems, such as the Fatou lemma, the dominated convergence theorem, and the bounded convergence theorem.

In 1974, Sugeno [32] introduced the notion of fuzzy measure and fuzzy integral, which are now generalized to the nonadditive measure and the Sugeno integral, to evaluate nonadditive or nonlinear quality in systems engineering. A nonadditive measure is a set function that is monotonic and vanishes at the empty set. This type of set function,

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in conjunction with several types of nonlinear integrals, is closely related to the theory of capacities and imprecise probabilities and has been extensively studied with applications to decision theory under uncertainty, game theory, data mining, some economic topics under Knightian uncertainty and others [4,5,9,19–21,37,38].

In order to put nonlinear integrals into practical use, it is inevitable to formulate some limit theorems assuring that the limit of the integrals of a sequence of functions is the integral of the limit function. Many attempts have thus been made to formulate the monotone convergence theorem for nonlinear integrals such as the Choquet integral [4], the Šipoš integral [27], the Sugeno integral [32], also known as the fuzzy integral [22], and the Shilkret integral [26], also known as the (N) fuzzy integral [43]; see [1,3,7,16–18,22,23,27,29–31,35,36,40–45] among others. For a recent survey of convergence theorems for the Choquet and the Sugeno integral we refer the reader to [11], which also contains fundamental convergence theorems of measurable functions such as the Lebesgue, the Riesz, the Egorov, and the Lusin theorem in nonadditive measure theory. However, to the best of knowledge, there is no unified approach to limit theorems in literature that are simultaneously applicable to both the Lebesgue integral as a linear integral and the Choquet, the Šipoš, the Sugeno, and the Shilkret integral as nonlinear integrals.

The purpose of the paper is to present a unified approach to the monotone convergence theorem for nonlinear integrals. A nonlinear integral may be viewed as a nonlinear functional $I\colon \mathcal{M}(X) \times \mathcal{F}^+(X) \to [0,\infty]$, where X is a non-empty set, $\mathcal{A}$ is a field of subsets of X, $\mathcal{M}(X)$ is the set of all nonadditive measures on $(X,\mathcal{A})$, and $\mathcal{F}^+(X)$ is the set of all $\mathcal{A}$-measurable non-negative functions on X. We thus formulate our general type of monotone convergence theorem for such a functional. The key tool is a perturbation of functional that manages not only the monotonicity of the functional but also the small change of the functional value $I(\mu, f)$ arising as a result of adding small amounts to the measure μ and the function f in the domain of I. This tool was already introduced and utilized as a means for formulating a bounded convergence in measure theorem and a portmanteau theorem in nonadditive measure theory [12,13]. It is worth mentioning that our approach is also applicable to the Lebesgue integral when the nonadditive measure μ is σ-additive.

The paper is organized as follows. In Section 2 we recall some definitions on nonadditive measures and nonlinear integrals. In Section 3 we introduce some classes of integral functionals, in particular, the class of perturbative functionals. We also show that not just the Lebesgue integral but also the Choquet, the Šipoš, the Sugeno, and the Shilkret integral belong to those classes. In Section 4 we discuss the MICT for integral functionals and show that, as is the case with the Lebesgue integral, the functional convergence $I(\mu, f_n) \uparrow I(\mu, f)$ is equivalent to the continuity of μ from below and the inner regularity of I, that is, $I(\mu, f) = \sup\{I(\mu, h)\colon h \text{ is simple and } 0 \le h \le f\}$ for every $f \in \mathcal{F}^+(X)$. In Section 5 we turn to the MDCT. Unlike the case of the Lebesgue integral, the MDCT does not follow from the MICT in general for lack of linearity of integral. In fact, we show that the functional convergence $I(\mu, f_n) \downarrow I(\mu, f)$ is equivalent to the continuity of μ from above and the μ-truncated property of the decreasing sequence $\{f_n\}_{n\in\mathbb{N}}$ for I, that is, for any $\varepsilon > 0$, there is $c > 0$ such that $I(\mu, f_n) - \varepsilon \le I(\mu, f_n \wedge c)$ for all $n \in \mathbb{N}$.

The MICT and MDCT for the Choquet, the Šipoš, the Sugeno, and the Shilkret integral are given in Section 6 as applications of our functional form of the monotone convergence theorems. Section 7 makes concluding remarks about the extension to the symmetric and the asymmetric integrals, other limit theorems, and the comparison with already proposed unified approach to limit theorems. Section 8 provides some of the proofs in Sections 3 and 5.

2. Preliminaries

In this paper, unless stated otherwise, X is a non-empty set and $\mathcal{A}$ is a field of subsets of X. Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of all natural numbers, respectively. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ with usual total order. For any $a, b \in \overline{\mathbb{R}}$ and any functions $f, g\colon X \to \overline{\mathbb{R}}$, let $a \vee b := \max(a,b)$ and $a \wedge b := \min(a,b)$ and let $(f \vee g)(x) := f(x) \vee g(x)$ and $(f \wedge g)(x) := f(x) \wedge g(x)$ for every $x \in X$. If $A \subset \mathbb{R}$ is non-empty and not bounded from above (below) in $\mathbb{R}$, let $\sup A := \infty$ ($\inf A := -\infty$). With this convention, every non-empty subset of $\overline{\mathbb{R}}$ has a supremum and an infimum in $\overline{\mathbb{R}}$. We adopt the usual conventions for algebraic operations on $\overline{\mathbb{R}}$. We also adopt the convention $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$ and $\inf \emptyset = \infty$. Instead of an ambiguous expression $c > 0$, we explicitly write $c \in (0,\infty]$ if the positive number c may take ∞. In other words, $c > 0$ always means $c \in (0,\infty)$. We use this notational convention for similar cases.

A function $f : X \to [-\infty, \infty]$ is said to be $\mathcal{A}$-*measurable* if $\{f \ge t\} := \{x \in X\colon f(x) \ge t\} \in \mathcal{A}$ and $\{f > t\} := \{x \in X\colon f(x) > t\} \in \mathcal{A}$ for every $t \in \overline{\mathbb{R}}$. If f and g are $\mathcal{A}$-measurable and $c \in \mathbb{R}$, then so are $f^+ := f \vee 0$, $f^- := (-f) \vee 0$, $|f| := f \vee (-f)$, cf, $f + c$, $(f - c)^+$, $f \vee g$, and $f \wedge g$. Note that $f = f \wedge c + (f - c)^+$. Let $\mathcal{F}(X)$