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When upper conditional probabilities are conditional possibility measures

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Abstract

Conditioning for (non-additive) uncertainty measures is still an open problem. This is essentially due to the fact that these measures can be viewed either as lower or upper probabilities or axiomatically defined, independently of probability theory. Focusing on possibility measures on finite domains, the aim is to generalize to the conditional case the interpretation of possibility measures as upper envelopes of extensions of suitable probability measures.

Keywords: Generalized Bayesian rule; Conditional possibility; Conditional probability; Inference; Upper envelope; Weak logical independence

1. Introduction

Probability theory has a fundamental role for dealing with uncertainty, even if sometimes the decision maker has only an incomplete, partial, imprecise probabilistic information. In such situations it is unavoidable to consider a set of probabilities together with the corresponding envelopes, which turn out to be non-additive uncertainty measures (see, e.g., [12,20,21,27,42]).

For example, prior probabilities are crucial in Bayesian statistics, even if sometimes the elicitation of the prior is not an easy task. For this, in some applications (for instance, in decision problems with multiple agents [15], in the reconstruction of images or when misclassified variables are present [37]) instead of assessing a single prior probability, one can select a suitable family of priors.

Concerning the relationship between probability theory and the chosen family of non-additive measures, it is well-known that the upper [lower] envelope of the coherent extensions of a coherent probability is subadditive [superadditive] but is not necessarily 2-alternating [2-monotone] (see, for instance, [13,42] and Example 1 in [2]).

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Nevertheless, starting from a probability on an algebra the upper [lower] envelope of its extensions to another algebra is a plausibility [belief] function (see [9,18,20,22,28,42]). In [25] it is provided a characteristic property for an upper probability to coincide with a possibility measure, which led in [10–12] to characterize possibility measures as upper envelopes of the extensions of a probability when a logical condition (*weak logical independence*) holds among the involved algebras.

Starting from the seminal works [24,46], possibility measures have received a lot of attention, due to their key role in fuzzy set theory, soft computing and decision theory. The debate on the conditioning for possibility measure or, more generally, for non-additive uncertainty measures is still an open issue, even if several conditioning notions have been introduced in literature (see, e.g., [1,7,14,16,20,24,26,30]).

A first aspect in which the given conditioning notions differ is how the conditional measure is defined: essentially, either starting from a non-additive unconditional measure through a suitable operation or *directly* as a function satisfying a set of axioms. Furthermore, conditional uncertainty measures can be viewed as specific lower or upper conditional probabilities, so the relation with probability theory can be investigated. In this line, in [5] we have generalized the Bayesian rule of conditioning for belief functions on finite domains, originally proposed by [20,28,32].

In this paper we go further in this investigation by studying when conditional plausibility functions are conditional possibility measures. In detail, focusing on a finite Boolean algebra $\mathcal{A}'$, we introduce the notion of *Bayesian conditional possibility* (or *B-conditional possibility* for short) which is obtained through a generalization of the Bayesian conditioning rule [20], relying on a linearly ordered class of possibility measures. For any B-conditional possibility measure $\Pi_B(\cdot|\cdot)$ defined on $\mathcal{A}' \times \mathcal{A}'^0$, with $\mathcal{A}'^0 = \mathcal{A}' \setminus \{\emptyset\}$, there exists a finite Boolean algebra $\mathcal{A}$ and a conditional probability $P(\cdot|\cdot)$ defined on $\mathcal{A} \times \mathcal{A}^0$ such that the upper envelope of its extensions coincides with $\Pi_B(\cdot|\cdot)$ on $\mathcal{A}' \times \mathcal{A}'^0$. Furthermore, we investigate the conditions under which the upper envelope of the extensions of a full conditional probability is a full B-conditional possibility. It is shown that weak logical independence of two finite algebras $\mathcal{A}$ and $\mathcal{A}'$, is a sufficient (but not necessary) condition for the upper envelope of the conditional probabilities extending a conditional probability $P(\cdot|\cdot)$ defined on $\mathcal{A} \times \mathcal{A}^0$ to be a B-conditional possibility on $\mathcal{A}' \times \mathcal{A}'^0$. Contrary to the unconditional case, there are B-conditional possibilities $\Pi_B(\cdot|\cdot)$ on $\mathcal{A}' \times \mathcal{A}'^0$ for which there is no finite algebra $\mathcal{A}$, weakly logically independent of $\mathcal{A}'$, such that a conditional probability on $\mathcal{A} \times \mathcal{A}^0$ gives rise to $\Pi_B(\cdot|\cdot)$ as upper envelope of its extensions (see Example 6). Thus we introduce a weaker form of weak logical independence that allows us to provide a necessary and sufficient condition for the upper envelope of the extensions of a conditional probability $P(\cdot|\cdot)$ defined on $\mathcal{A} \times \mathcal{A}^0$ to be a B-conditional possibility measure on $\mathcal{A}' \times \mathcal{A}'^0$.

The paper is structured as follows. In Section 2 some preliminary notions on conditional probabilities and their extensions are recalled, while in Section 3 the problem of conditioning for possibility measures is introduced. Section 4 presents the generalized Bayesian conditioning rule which gives rise to full B-conditional possibilities. Section 5 copes with the logical notion called weak logical independence. Such condition implies a conditional form of nesting giving rise to a full B-conditional possibility as the upper envelope of the extensions of any full conditional probability. Since weak logical independence turns out to be only a sufficient condition for obtaining a full B-conditional possibility, in Section 6 a necessary and sufficient condition is given. Finally, in Section 7 some conclusions are drawn.

2. Conditional probability and its enlargement

Let $\mathcal{A}$ be a Boolean algebra of *events* E's and denote by $(\cdot)^c$, $\vee$ and $\wedge$ the usual Boolean operations of contrary, disjunction and conjunction, respectively. The *sure event* Ω and the *impossible event* $\emptyset$ coincide, respectively, with the top and bottom elements of $\mathcal{A}$ endowed with the partial order $\subseteq$ of *implication*.

A *conditional event* $E|H$ is an ordered pair of events with $H \neq \emptyset$, where the conditional event $E|\Omega$ is customarily identified with E. Let $\mathcal{H}$ be an *additive class* (i.e., a set of events closed under finite disjunctions) such that $\mathcal{H} \subseteq \mathcal{A}^0 = \mathcal{A} \setminus \{\emptyset\}$. An arbitrary set of conditional events $\mathcal{G} = \{E_i|H_i\}_{i\in I}$ can always be embedded into a minimal structured set $\mathcal{A} \times \mathcal{H}$, where $\mathcal{A} = \langle\{E_i, H_i\}_{i\in I}\rangle$ is the Boolean algebra generated by $\{E_i, H_i\}_{i\in I}$ and $\mathcal{H}$ is obtained closing $\{H_i\}_{i\in I}$ under finite disjunctions. In the following such set $\mathcal{A} \times \mathcal{H}$ is denoted as $\langle\langle\mathcal{G}\rangle\rangle$. Let us recall the relation of implication for conditional events introduced by Goodman and Nguyen in [31]:

$$E|H \subseteq_{GN} F|K \iff \begin{cases} E \wedge H \subseteq F \wedge K, \\ E^c \wedge H \supseteq F^c \wedge K. \end{cases}$$