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On maxitive integration

Marco E.G.V. Cattaneo

Department of Mathematics, University of Hull, United Kingdom

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Abstract

A functional is said to be maxitive if it commutes with the (pointwise) supremum operation. Such functionals find application in particular in decision theory and related fields. In the present paper, maxitive functionals are characterized as integrals with respect to maxitive measures (also known as possibility measures or idempotent measures). These maxitive integrals are then compared with the usual additive and nonadditive integrals on the basis of some important properties, such as convexity, subadditivity, and the law of iterated expectations.

Keywords: Maxitive measures; Nonadditive integrals; Choquet integral; Convexity; Subadditivity; Law of iterated expectations

1. Introduction

The standard integral of a function with respect to a probability measure (i.e., its expectation) can be interpreted as a weighted average of the function values. Such averages play a central role in the theory of decision making under risk, in which decisions are selected by maximizing expected utility or minimizing expected loss. However, in practice decisions are often selected on the basis of best-case or worst-case evaluations. Such evaluations are described by integrals that are maxitive (i.e., the integral of a pointwise maximum of functions is the maximum of their integrals), while the standard integral is additive (i.e., the integral of a pointwise sum of functions is the sum of their integrals). More generally, maxitive integrals appear as aggregation functionals in many fields of application, such as operations research, information fusion, or control theory (see for example [19,32,6]).

Maxitive integrals can be seen as extensions of maxitive measures (i.e., the measure of a union of sets is the maximum of their measures). These nonadditive measures play a central role in possibility theory and tropical or idempotent mathematics (they are also called possibility measures or idempotent measures), but have been studied also in other contexts (see for instance [17,22,34]). Contrary to the case of additivity, the countable (or even uncountable) maxitivity of measures and integrals does not pose particular difficulties, and measurability restrictions are thus unnecessary. This is discussed in the next section.

E-mail address: m.cattaneo@hull.ac.uk.

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The rest of the paper can be divided into two parts. The first part (corresponding to Sections 3–5) studies the integral of nonnegative functions that was introduced by Shilkret [30] and is characterized by maxitivity and positive homogeneity. Some of its properties, such as subadditivity and the law of iterated expectations, are derived and compared with corresponding properties of alternative definitions of integral. In particular, new results about the Choquet integral [11,15] are also presented: necessary and sufficient conditions for countable comonotonic additivity and for the law of iterated expectations.

The second part corresponds to Section 6 and pursues the definition of a maxitive integral for all real functions. In fact, the Shilkret integral cannot be extended in a satisfactory way to a maxitive integral of real functions. Instead, the integral characterized by maxitivity and additive homogeneity is introduced and named convex integral. The reason for the name is its convexity, which is derived besides other properties, such as the law of iterated expectations. The convex integral is strictly related to the idempotent integral of tropical or idempotent mathematics [22] and to convex measures of risk [18].

2. Maxitive measures

Let Ω be a nonempty set, and let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a collection of subsets of Ω. When κ is a cardinal, $\mathcal{A}$ is said to be closed under κ-union if and only if $\bigcup \mathcal{B} \in \mathcal{A}$ for all nonempty $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| \leq \kappa$. An extended real-valued set function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ is said to be κ-maxitive if and only if $\mu(\bigcup \mathcal{B}) = \sup_{A \in \mathcal{B}} \mu(A)$ for all nonempty $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| \leq \kappa$ and $\bigcup \mathcal{B} \in \mathcal{A}$. Furthermore, μ is said to be monotonic if and only if $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{A}$ such that $A \subseteq B$. Monotonicity is implied by finite maxitivity (i.e., 2-maxitivity), while countable maxitivity (i.e., $\aleph_0$-maxitivity) was assumed by Shilkret [30] in his definition of maxitive measures.

Theorem 1. *Let κ be a cardinal, and let $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ be κ-maxitive. If $\mathcal{A}$ is closed under κ-union, or $\mathcal{A}$ is a ring (i.e., closed under finite union and finite intersection), then there is a κ-maxitive extension of μ to $\mathcal{P}(\Omega)$.*

Proof. The cases with $\kappa \leq 1$ or $\mathcal{A} = \varnothing$ are trivial, so assume $\kappa > 1$ and $\mathcal{A} \neq \varnothing$. Consider first the case with $\mathcal{A}$ closed under κ-union, and define $\mu' : A \mapsto \inf_{B \in \mathcal{A} : A \subseteq B} \mu(B)$ on $\mathcal{P}(\Omega)$, where $\inf \varnothing = +\infty$. Since μ is monotonic, μ' is a monotonic extension of μ. Therefore, in order to prove the κ-maxitivity of μ', it suffices to show $\mu'(\bigcup \mathcal{B}) \leq \sup_{A \in \mathcal{B}} \mu'(A)$ for all nonempty $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ such that $|\mathcal{B}| \leq \kappa$ and $\sup_{A \in \mathcal{B}} \mu'(A) < +\infty$. When $\mathcal{B}$ is such a set, for each $A \in \mathcal{B}$ and each $\varepsilon \in \mathbb{R}_{>0}$ there is an $A_\varepsilon \in \mathcal{A}$ such that $A \subseteq A_\varepsilon$ and $\mu(A_\varepsilon) < \mu'(A) + \varepsilon$, and thus

$$\mu'\left(\bigcup \mathcal{B}\right) \leq \inf_{\varepsilon \in \mathbb{R}_{>0}} \mu\left(\bigcup_{A \in \mathcal{B}} A_\varepsilon\right) \leq \inf_{\varepsilon \in \mathbb{R}_{>0}} \sup_{A \in \mathcal{B}} \left(\mu'(A) + \varepsilon\right) = \sup_{A \in \mathcal{B}} \mu'(A).$$

Now let $\mathcal{A}$ be a ring and let κ be infinite (the case with finite κ has already been considered above). If $\mathcal{B}, \mathcal{B}' \subseteq \mathcal{A}$ are two nonempty sets with cardinality at most κ, and $\bigcup \mathcal{B} = \bigcup \mathcal{B}'$, then

$$\sup_{A \in \mathcal{B}} \mu(A) = \sup_{A \in \mathcal{B}} \mu\left(\bigcup_{B \in \mathcal{B}'} (A \cap B)\right) = \sup_{A \in \mathcal{B}} \sup_{B \in \mathcal{B}'} \mu(A \cap B) = \sup_{B \in \mathcal{B}'} \mu(B).$$

Hence, the function $\tilde{\mu} : \bigcup \mathcal{B} \mapsto \sup_{A \in \mathcal{B}} \mu(A)$ on $\tilde{\mathcal{A}} = \left\{\bigcup \mathcal{B} : \mathcal{B} \subseteq \mathcal{A}, 0 < |\mathcal{B}| \leq \kappa\right\}$ is a well-defined extension of μ. In order to complete the proof of the theorem, it suffices to show that $\tilde{\mathcal{A}}$ is closed under κ-union and $\tilde{\mu}$ is κ-maxitive, and this follows easily from the fact that (assuming the axiom of choice) the product of two infinite cardinals is their maximum. □

Since the focus of the present paper is on maxitive measures and integrals, Theorem 1 implies that measurability restrictions are unnecessary, and $\mathcal{A} = \mathcal{P}(\Omega)$ can always be assumed. Moreover, in order to simplify the following results, only nonnegative real-valued set functions $\mu : \mathcal{P}(\Omega) \to \mathbb{R}_{\geq 0}$ are considered, and without real loss of generality $\mu(\Omega) = 1$ is imposed. A set function $\mu : \mathcal{P}(\Omega) \to \overline{\mathbb{R}}$ is said to be a capacity on Ω (in the most general sense) if and only if μ is monotonic, $\mu(\varnothing) = 0$, and $\mu(\Omega) = 1$.

Capacities μ on Ω are often used in applications as quantitative descriptions of uncertain belief or information about $\omega \in \Omega$. The larger the value $\mu(A)$, the larger the plausibility of $\omega \in A$, or the larger the implausibility of $\omega \notin A$.

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