ARTICLE IN PRESS
FSS:6990

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems ••• (••••) •••–•••

FUZZY sets and systems

www.elsevier.com/locate/fss

Regularity aspects of non-additive set multifunctions

Alina Gavriluţ *, Gabriela Apreutesei

Faculty of Mathematics, Al.I. Cuza University, Bd. Carol I, No. 11, Iaşi, 700506, Romania

Received 1 April 2015; received in revised form 29 January 2016; accepted 4 February 2016

Abstract

In this paper, continuity properties of set multifunctions, such as regularity and continuity from above and below (as well as others), are introduced by way of a Wijsman topology. Some of the relationships between them are established. Different examples, counterexamples and applications are provided.

Keywords: Wijsman topology; Monotone set multifunction; Multisubmeasure; Regularity; Exhaustive; Order continuous; Continuous from below; Continuous from above

1. Introduction

Non-additive measures can be used for modeling problems in nondeterministic environments. In recent years, this area has been widely developed and a variety of topics have been investigated.

Due to their various applications in economics, games theory, artificial intelligence and other important fields, many considerations from measure theory have been extended to multifunctions (in particular, integrability and continuity properties [6,12,25]). Also, results from nonadditive measures theory have been generalized to the set-valued case (see Guo and Zhang [14] for Kuratowski convergence and Gavrilut [12–15] for Hausdorff topology). It is our aim to investigate regularity, a basic concept subject in Egoroff and Lusin's theorems, for the case of multifunctions.

A well-known method for investigating a multifunction is to endow the set of its values with a hypertopology (for instance: Hausdorff, Vietoris, Wijsman, Fell, Attouch–Wets). Because of its numerous applications, the study of hypertopologies has become of great interest. Interesting results were obtained by Lorenzo and Maio [23] in melodic similarity, Lu et al. [24] in word image matching, Kunze et al. [17] and Wicks [27] in self-similarity and fractality. Important results concerning the Hausdorff and Vietoris topologies can be found in Beer [3–5], Apreutesei [1,2], Hu and Papageorgiou [16], etc.

Wijsman considered in [24] the weak topology on the collection of nonempty closed subsets of a metric space generated by the distance functionals (viewed as functions of a set argument). Researchers in hypertopologies found this topology quite useful, and subsequently a vast body of literature developed on this topic (see [1,3–5,7,9,15,18,

* Corresponding author.
E-mail addresses: gavrilut@uaic.ro (A. Gavriluţ), gapreutesei@yahoo.com (G. Apreutesei).

http://dx.doi.org/10.1016/j.fss.2016.02.003
0165-0114/© 2016 Published by Elsevier B.V.

Please cite this article in press as: A. Gavriluţ, G. Apreutesei, Regularity aspects of non-additive set multifunctions, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.02.003

24,25], etc.). Intending to choose a convenient topology for the set of values of the studied multifunctions, in this investigation we prefer to consider the Wijsman topology [30] instead of the Hausdorff or Vietoris topologies, as it better describes the pointwise properties. For instance, in some examples of fractals [29], such as neural networks and circulatory systems, the uniform property of the Hausdorff topology is inappropriate. Therefore, this paper studies the regularity properties of nonadditive multifunctions with respect to the Wijsman topology. The definition and basic properties of this hypertopology are given in Section 2.

In process modeling, the additivity and sub-additivity of the measure are too restrictive. The main difficulty when extending some classical results from measure theory to the case of nonadditive set multifunctions, is to "simulate" the countable additivity using weaker conditions (see Pap [25,26]). In this way we arrived at some notions of the continuity of the multimeasure: from above, from below, o-continuity and exhaustivity (which, in reality, coincides with the boundedness condition). In Section 3 we propose a variant of these definitions according to the Wijsman topology. There exist notable differences between the univocal and the multivalued case. For instance, as is well known, for real-valued finitely additive set functions, countable additivity and continuity from above/below are equivalent. Also, for real-valued submeasures and for multisubmeasures of finite variation, taking values in the family of all nonvoid closed bounded subsets of a Banach space, order continuity is equivalent to countable subadditivity. Many other results concerning relationships among order continuity, continuity from above/below and exhaustivity are not usually valid for general nonadditive set (multi)functions. The relationships among them in the case of the Wijsman topology are formulated in the first subsection of Section 3 (see Proposition 3.1.2, Theorems 3.1.3, 3.1.6, 3.1.8 and their Corollaries). In the second subsection of Section 3 one proves (by way of several examples) that these types of continuity are consistent and different from each other, and that they do not coincide with similar notions with respect to the Hausdorff topology.

Regularity is an important property of continuity, and connects measure theory and topology. It gives us an important tool to approximate general Borel sets by more tractable sets such as, for instance, compact and/or open sets. Important theorems in measure theory, like Egoroff or Lusin theorems, use in their proofs the regularity of the (multi)measure (see [10–13,19–22]). Section 4 introduces the notion of regularity for a multifunction with respect to the Wijsman topology and gives some characteristic properties (Proposition 4.6, Theorem 4.7 and Corollary 4.8). The link between regularity and σ-additivity proved to be quite strong (see the Baire measures for example). For this reason we investigated first the above types of continuities for multifunctions in order to get new regularity results. In Theorem 4.9 we shall prove that the continuities defined in the Wijsman topology lead to necessary conditions for regularity. This proves that our definitions are fruitful in the Wijsman sense.

Let us mention the following: in [12], we dealt with the stronger "subadditivity" condition of the set multifunction (with respect to the inclusion of sets and the Minkowski addition), and also with certain weaker types of regularity in the Wijsman topology than the ones introduced here. More precisely, Definitions 2.1–2.4 [12] introduced certain sorts of regularity: since the subbase for the Wijsman topology is composed of two types of collections of sets (see the definition of $\mathcal{F}$ in Section 2 of the present paper), we had corresponding types of regularity with respect to each collection of sets. Accordingly, practically speaking, if $\mu(A) \in \mathcal{F}$, then for any B which approximates A (with compact sets from the left, open sets from the right or both simultaneously), we have $\mu(B) \in \mathcal{F}$ $(*)$. For instance, in the simultaneous approximation from the left and from the right for A, if $K \subset A \subset D$, then for any B with $K \subset B \subset D$, the condition $(*)$ is fulfilled. Moreover, several relationships between regularities (Theorems 2.5–2.6) as well as some correlations with other corresponding regularities in Vietoris, i.e. Hausdorff topologies (Theorems 2.7–2.10), were obtained in [12].

In the current paper, we renounce on the stronger hypothesis of subadditivity, and use stronger types of regularity. In short, this idea is described by the following: if $K \subset A \subset D$, then for any $B \subset D \setminus K$, $\mu(B)$ approximates $\mu(\emptyset)$.

All these conditions enable us to obtain certain special results:

1. We establish the relationships between the new types of regularity introduced in this paper.

2. These new types of regularity (viewed here as a continuity (and not just an approximation) regularity, as in [12]) are compared with other types of continuity properties: order continuity, continuity from below/above, exhaustivity etc. In this sense, different examples and counterexamples are provided.

3. The comparison with the corresponding types of continuities in Hausdorff topology are also highlighted through examples and counterexamples.

All these show that the results from this paper cannot been obtained in [10–13], as the approach for obtaining them is not similar.

Please cite this article in press as: A. Gavriluţ, G. Apreutesei, Regularity aspects of non-additive set multifunctions, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.02.003