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On cardinalities of finite interval-valued hesitant fuzzy sets

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ABSTRACT

Certain extensions of the classical fuzzy sets have been studied in depth since they have a remarkable importance in many practical situations. We focus on finite interval-valued hesitant fuzzy sets, as they generalize the most usual sets (fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets), so the results obtained can be immediately adapted to these types of sets. In addition, their membership functions are much more manageable than type-2 fuzzy sets.

In this work, the cardinality of finite interval-valued hesitant fuzzy sets is studied from an axiomatic point of view, together with several properties that this definition satisfies, which enable to relate it to the classical definitions of cardinality given by other authors for fuzzy sets.

1. Introduction

The fuzzy sets theory introduced by L. Zadeh in 1971 (see [28]) became an important generalization of the classical set theory, and has been deeply studied since then. Works in different fields have been developed around the fuzzy logic, such as image processing (see [2,16]) or privacy protection (see [14,15]).

A fuzzy set is characterized by its membership function. This function is usually determined by an expert, which can cause a kind of ambiguity. In order to deal with this issue, different generalizations of the fuzzy set theory have been developed. The interval-valued fuzzy sets given by Sambuc in 1975 (see [19]) are one of the most studied generalizations. These sets are characterized by a membership function which asssigns an interval for each point instead of a single value. Another well known type is an intuitionistic fuzzy set, developed by Atanassov in 1986 (see [1]). These objects are defined by two mappings, the membership function and the non-membership function. Type-2 fuzzy sets are a deeper extension given by Zadeh in 1975 (see [29]), where the membership function is defined by another fuzzy set at each point.

However, working with type-2 fuzzy sets is not an easy task, so Torra intruduced hesitant fuzzy sets in 2009 (see [21,22]) as an intermediate type of fuzzy sets. They assign a subset of the interval [0, 1] to each element instead of a fuzzy set, which makes them more manageable than type-2 fuzzy sets. In fact, Grattan–Guinness introduced in 1976 this type of sets (see [9]) under the name of set-valued fuzzy sets. Nevertheless, unlike Grattan–Guinness, Torra provided functional definitions

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of union and intersection for them. This type of sets presents good properties which make them suitable for researching (see [3,27]), and specially, they are applicable in decision making (see [5,10,23]). They not only suit well some applications, but at the same time they can be handled well as mathematical objects. Several extensions of hesitant fuzzy sets have been defined lately (see [18]). In our work we study finite interval-valued hesitant fuzzy sets (see [12]), whose membership function assigns a union of a finite number of disjoint closed subintervals of [0, 1].

The concept of cardinality has always been interesting as counting is one of the basic elementary mathematical activities. For finite crisp sets, it is an intuitive concept which is easy to define mathematically. However, when working with fuzzy sets, this definition is not straightforward, and different options have been given by several authors. De Luca and Termini in 1972 (see [7]) defined the σ-count cardinality, which is defined by the sum of all the membership degrees. Ralescu proposed in 1995 (see [17]) the concepts of fuzzy and crisp cardinality, applied in other papers, such as [14], where it is used for the protection of privacy in microdata using fuzzy partitions. From an axiomatic point of view, Wygralak gave a definition of scalar cardinality for fuzzy sets in 2003 (see [24]), which includes the crisp cardinality given by Ralescu as it is proved in our paper. The axiomatic definition given by Wygralak has been extended to interval-valued fuzzy sets by Deschrijver and Král' (see [8]), among other results connected to this definition.

As it has been mentioned before, the hesitant fuzzy logic is a fresh theory. Furthermore, the extension which is taken into account in this paper, the finite interval-valued hesitant fuzzy sets, is even more recent, as it has been given for the first time in 2013 in [6]. The aim of this paper is to study the cardinality for this type of sets. The given definition is an axiomatic one, which allows to get different cardinalities, without the restrictions of a fixed one. A particular case is also studied, which presents good properties for fuzzy sets, as it matches the crisp cardinality given by Ralescu. Other results for the axiomatic definition of cardinality for finite interval-valued hesitant fuzzy sets are studied, and some properties of this cardinality are given.

The paper is set out as follows: Section 2 is split into two subsections with preliminary concepts about hesitant fuzzy sets and cardinality in a fuzzy environment, respectively. Section 3 is focused on the new definition of cardinality for finite interval-valued hesitant fuzzy sets and related results. Section 4 shows some examples of cardinalities. In Section 5 we propose an illustrative example and we compare our approach with some previous approaches, in order to see the advantages of the proposed method. Finally, in Section 6 the main conclusions of this work are highlighted.

2. Preliminaries

In this section the basic concepts required to follow the paper are introduced. Main concepts about fuzzy sets and their extensions are described in the first subsection. In the second subsection, hesitant fuzzy logic as well as its motivation are depicted. Finally, some well known definitions of cardinality for fuzzy sets are tackled.

2.1. Hesitant fuzzy sets

The initial concepts detailed in this subsection are well known and can be found in several sources, such as [11]. All these definitions are important to compare fuzzy and hesitant logics and to understand the motivation of the second one.

Fuzzy sets were first introduced by Zadeh (see [28]) as a set $A = \{(x, \mu_A(x)) | x \in X\}$, where X denotes a non-empty set and μ_A the membership function $\mu_A\colon X \to [0, 1]$. Along this paper, $FS(X)$ will denote the set of all fuzzy sets in X.

The concepts of intersection and union of fuzzy sets are needed along the paper. The standard definition is the one used in this paper, that is, the membership functions of the union of these two sets, $A \cup B$, and the intersection, $A \cap B$, respectively, are: $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$ and $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$.

Determining the membership degree sometimes represents a bottleneck because it is hard to define it accurately by the experts. Different types of fuzzy sets have been introduced to make softer this decision. One of them is the notion of interval-valued fuzzy sets. These sets (originally developed by Sambuc in [19]) constitute a generalization of the fuzzy sets, where each point is associated to an interval instead to a just one value. Type-2 fuzzy sets are another generalization of the fuzzy sets developed by Zadeh (see [29]). In this case, the membership function is a map into $FS([0, 1])$. However, the membership function of a type-2 fuzzy set is hard to handle. Hesitant fuzzy sets represent an intermediate step between fuzzy sets and type-2 fuzzy sets. Due to the good properties of the membership functions of these new sets compared to the ones of type-2 fuzzy sets, they represent an interesting alternative as a generalization of fuzzy sets. Furthermore, all the results obtained in a hesitant environment can be quickly adapted to other types of sets, such as interval-valued fuzzy sets and the classical fuzzy sets, since they are a generalization of them.

Hesitant fuzzy logic was recently defined by Torra in [21,22], although the first introduction of this concept was made by Grattan–Guinnes in [9], under the name of set-valued fuzzy set. Nevertheless, unlike Grattan–Guinness, Torra provided functional definitions of union and intersection for them. Several papers related to this logic have been published, such as [3], where the basic concepts can be found.

A possible justifcation of a hesitant fuzzy set could be the case when several experts estimate the membership of a particular point to a collection. Moreover, we assume, that the evaluators (or at least some of them) are aware of the others' evaluation, so their decisions are not independent, hence it is reasonable to consider them in a single object – a hesitant fuzzy set, rather than in a collection of individual fuzzy sets. This point of view is very similar to the logical background of