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Fuzzy theoretic approach to signals and systems: Static systems☆

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ABSTRACT

"*Fuzzy Theoretic Approach to Signals and Systems*" assumes all system variables and parameters as uncertain (i.e. being characterized by membership functions), develops a mathematical theory for analytically determining the membership functions on system variables and parameters, derives algorithms for estimating the parameters of membership functions, and establishes robustness and convergence properties of the estimation algorithms. The membership functions are analytically determined via solving a variational optimization problem that maximizes the "*over-uncertainties-averaged-log-membership*" of the observed data around an initial guess. This paper develops the analytical fuzzy theory for the particular case of a multi-input single-output static system affected by noises. The theory facilitates designing an adaptive filtering algorithm. The robustness of the adaptive filtering algorithm is proved theoretically via a mathematical analysis. Numerical experiments further demonstrate the robustness of the filtering algorithm. A comparison of the algorithm with the state-of-art methods is made using the practical biomedical applications related to the modeling and analysis of heart rate signals for assessing the physiological state of an individual.

1. Introduction

The data-driven models capture regularities in the data. The learning of a model is the process of adjusting the flexible model parameters to best explain the observed data. The observed data would inevitably possess a certain degree of *uncertainty* regarding the representation of the true process. The uncertainties, if not taken care of, would lead to erroneously learned models. The aim of this study is to formalize the idea of applying membership functions to take care of uncertainties during the learning of data models. The modeling methods fit the observed input-output data (x, y) through adjustment of model parameters α while taking into consideration the unobserved (*latent* or *hidden*) variables h. Specifically, the central

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theme of this study is to determine *optimal* membership functions on observed data (x, y), unknown model parameters α, and unobserved (obviously unknown) variables h.

Despite the fuzzy theory being a densely studied subject, most (if not all) fuzzy researchers have to rely on numerical methods for the design of fuzzy systems. Several previous studies have focused on the estimation theory of linear parameters of Takagi-Sugeno type fuzzy models while relying on ad-hoc numerical algorithms for determining membership functions. This is understandable as the design criterion is of optimizing a certain objective functional which due to nonlinearity can't be analytically optimized with respect to membership function parameters. Those studies have applied the mathematical tools of system and control theory for linear fuzzy model parameters [9–11,15–18]. The numerical algorithms extensively used in literature for designing fuzzy systems includes multi-objective evolutionary [1,2,7,8,24,25] and data clustering algorithms [5,6,19,22]. On the other hand, treating the fuzzy model parameters as random variables, the concept of variational Bayes borrowed from probability theory has been applied to design stochastic fuzzy systems [12–14]. The author in [4] has also suggested a fuzzy approach to signal theory in parallel to the probabilistic approach via incorporating uncertainty in the description of mathematical transformations typically used for signal analysis. However, our focus is on the learning aspect of an uncertain signal model and not on introducing new fuzzy based mathematical transformations to extract signal features. While the research presented in [14] provides a mathematical framework to design both fuzzy filtering and analysis algorithms in a unified manner, the developed theory being built up for stochastic fuzzy models contributes more to the probability theory than the fuzzy theory. It is worth mentioning that attempts have been made to apply the tools and concepts such as $H^{\infty}-$filtering [26], Kalman filtering [21], and information-theoretic measures [3,20] for determining membership functions. However, a "*self-contained*" theory on the optimization of membership functions was not available. A large literature (see, e.g. [27–29]) is available in the fuzzy control area where a fuzzy mixture of linear systems is used to approximate the nonlinear system with the aim of achieving robustness and stability of the filtering errors. Our problem is different from the fuzzy filtering problem extensively studied in fuzzy control area as our aim is to identify the unknown process model. We observe that the development of powerful analytical methods for the inclusion of uncertainties into mathematical modeling and analysis of data has been a challenge for the fuzzy research community.

Our research group develops the mathematical tools to facilitate the design and analysis of membership functions which take care of the uncertainties affecting signals and systems. This contributes to the so-called "*Fuzzy Theoretic Approach to Signals and Systems*". "*Fuzzy Theoretic Approach to Signals and Systems*" assumes all system variables and parameters as uncertain (i.e. being characterized by membership functions), develops a mathematical theory for analytically determining the membership functions on system variables and parameters, derives algorithms for estimating the parameters of membership functions, and establishes robustness and convergence properties of the estimation algorithms. Our approach to fuzzy theory development, for a given modeling scenario, consists of following steps:

1. An initial guess is made regarding the functional representation of multivariate membership functions on model parameters (α) and hidden variables (h). The initial guess regarding the membership functions on α and h is represented as $\mu(\alpha)$ and $\mu(h)$ respectively.
2. A mathematical expression of the joint membership function on input-output data (x, y) is derived taking into consideration the structure of the model and the relationships among variables. The joint membership function on input-output data, being represented as $\mu(x, y; h, \alpha)$, is a function of h and α.
3. The average value of a function $f(x)$ with respect to x, where x is an uncertain variable with $\mu(x)$ as the membership function to characterize its uncertainty, is calculated as

$$\langle f(x)\rangle_{\mu(x)} = \frac{1}{\int \partial x\, \mu(x)} \int \partial x\, \mu(x) f(x).$$

The membership functions on α and h are determined by solving following problems:

(a) The averaged logarithmic value of $\mu(x, y; h, \alpha)$ is maximized with respect to arbitrary membership functions having fixed integrals (i.e. with fixed area under their curves). Mathematically,

$$\{q^*(\alpha; k_\alpha), q^*(h; k_h)\} = \arg\max_{q(\alpha),q(h)} \left\{ \langle \log(\mu(x, y; h, \alpha))\rangle_{q(h)q(\alpha)} - \left\langle \log\left(\frac{q(\alpha)}{\mu(\alpha)}\right)\right\rangle_{q(\alpha)} - \left\langle \log\left(\frac{q(h)}{\mu(h)}\right)\right\rangle_{q(h)} \right\}$$

subjected to the fixed integral constraints:

$$\int \partial\alpha\, q(\alpha) = k_\alpha > 0, \quad \int \partial h\, q(h) = k_h > 0.$$

Here, $q(\alpha)$ and $q(h)$ represent arbitrary membership functions on α and h respectively. The optimization process maximizes an objective functional which consists of three terms. The first term equals the average log-membership value, when average is taken over uncertain variables α and h with $q(\alpha)$ and $q(h)$ respectively as the membership functions for representing the uncertainties. The second and third terms of the objective functional regularize the solution towards the initial guess $(\mu(\alpha), \mu(h))$.

(b) Once the analytical expressions for $q^*(\alpha; k_\alpha)$ and $q^*(h; k_h)$ have been derived using variational optimization technique, the values of k_α and k_h are so chosen such that maximum value of $q^*(\alpha; k_\alpha)$ and $q^*(h; k_h)$ w.r.t. α and h respectively