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Generalized qualitative Sugeno integrals

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ABSTRACT

Sugeno integrals are aggregation operations involving a criterion weighting scheme based on the use of set functions called capacities or fuzzy measures. In this paper, we define generalized versions of Sugeno integrals on totally ordered bounded chains, by extending the operation that combines the value of the capacity on each subset of criteria and the value of the utility function over elements of the subset. We show that the generalized concept of Sugeno integral splits into two functionals, one based on a general multiple-valued conjunction (we call integral) and one based on a general multiple-valued implication (we call cointegral). These fuzzy conjunction and implication connectives are related via a so-called semiduality property, involving an involutive negation. Sugeno integrals correspond to the case when the fuzzy conjunction is the minimum and the fuzzy implication is Kleene-Dienes implication, in which case integrals and cointegrals coincide. In this paper, we consider a very general class of fuzzy conjunction operations on a finite setting, that reduce to Boolean conjunctions on extreme values of the bounded chain, and are non-decreasing in each place, and the corresponding general class of implications (their semiduals). The merit of these new aggregation operators is to go beyond pure lattice polynomials, thus enhancing the expressive power of qualitative aggregation functions, especially as to the way an importance weight can affect a local rating of an object to be chosen.

1. Introduction

In the setting of Artificial Intelligence (for instance, recommender systems, cognitive robotics, but other fields as well), the use of decision rules based on numerical aggregation functions is not always natural. For instance, probabilities, utilities, importance weights cannot always be easily elicited from the user, by lack of time or lack of precision. Information systems advising persons cannot ask too many questions to users for modeling their preferences, nor collect from them meaningful numbers representing probabilities or criteria importance levels, or yet utility values. Even if they get them, making numerical operations on them looks debatable. Another example is when a referee has to fill a form to assess the merits of a paper for a journal, and numerical ratings are required. What is the precise meaning of these ratings? Does it make sense to compute averages from them?

In such situations it is more natural to resort to a qualitative approach to multicriteria evaluation. The rationale is to refrain from using numbers that look arbitrary or hard to collect, if we can evaluate decisions in a reasoned approach,

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without numerical calculations and address decision problems in the ordinal setting. There are two advantages: (i) a gain in robustness and the need for less data; (ii) qualitative methods lend themselves to a logical representation (which makes proposed choices more easily explainable). There are two possible choices of qualitative settings for representing notions such as utility ratings by several agents, importance levels and likelihood degrees:

- Use distinct non-commensurate scales. This makes the framework very restrictive as impossibility theorems regarding rational aggregation processes are often obtained (e.g., in voting theory).
- Use finite commensurate scales (taking advantage of notions facilitating commensurateness such as certainty equivalents) then one works with a finite ordered set of value classes.

In multi-criteria decision making, Sugeno integrals [39,40] are commonly used as qualitative aggregation functions [25] using finite scales and a commensurateness assumption between them. The definitions of these integrals are based on a monotonic set-function named capacity or fuzzy measure that aims to qualitatively represent the likelihood of sets of possible states of nature, the importance of sets of criteria, etc. These set functions are currently used in many areas such as uncertainty modeling [13,14], multiple criteria aggregation [3,23,24] or in game theory [37]. See also a recent book devoted to capacities in such areas [28]. Moreover, Sugeno integrals naturally lend themselves to a representation in terms of if-then rules involving thresholds [7,15,23], which makes them easy to interpret.

Capacities can be exploited in different ways when aggregating local ratings of objects according to various criteria, and different qualitative integrals (q-integrals, for short) can be obtained. In the case of Sugeno integrals, the capacity is used as a bound that restricts the global evaluation from below or from above. In other cases, the capacity is considered as a tolerance threshold such that overcoming it leads to improving the global evaluation of the object under study. When this threshold is not reached there are two possibilities. Either the local rating remains as it stands or it is modified: improved if the criterion is little important, downgraded if it is important.

These considerations give rise to new aggregation operations in the qualitative setting, such as soft and drastic integrals, studied in a recent paper [16]. Namely, we introduced variants of Sugeno integrals based on Gödel implication and its contrapositive version, using an involutive negation. It models qualitative aggregation methods that extend min and max, based on the idea of tolerance threshold beyond which a criterion is considered satisfied. These variants of integrals are valued in a scale equipped with a residuated implication and an involutive negation. More precisely, in [16], the evaluation scale is both a totally ordered Heyting algebra and a Kleene algebra. These new aggregation operations have been axiomatized in [17] in the setting of a complete bounded chain with an involutive negation.

In the present paper, which extends a conference paper [18], we try to cast this approach in a more general totally ordered algebraic setting, using multivalued conjunction operations that are not necessarily commutative, and implication operations induced from them by means of an involutive negation, with a view to preserve characterization theorems proved in [17] for Gödel implication and its contrapositive version, and non-commutative fuzzy conjunctions obtained via an involutive negation. We show that, once generalized in this way, the concept of Sugeno integral splits into two functionals, one based on a generalized conjunction (we call q-integral) and one based on an implication (we call q-cointegral). These fuzzy conjunction and implication connectives are related via a semiduality property, involving an involutive negation. Sugeno integrals correspond to the case when the fuzzy conjunction is the minimum and the implication is Kleene-Dienes implication, in which case integrals and cointegrals coincide.

The paper is structured as follows. Section 2 presents the main motivations for the study of new aggregation operations, namely the notion of weighted qualitative aggregation function. It studies the way importance weights of criteria affect local ratings of objects with respect to such criteria. Section 3 provides insights on the algebraic setting useful for generalizing Sugeno integrals, namely the possible dependence between fuzzy conjunctions and implications via residuation and a property called semiduality. Section 4 presents the proposed generalizations of Sugeno integrals, emphasizing the existence of distinct integrals and cointegrals respectively defined by means of fuzzy conjunctions and implications. Section 5 presents representation theorems for q-integrals and q-cointegrals.

2. Motivations

After recalling the weighted min and max aggregation functions, this section extends these aggregation functions to other weighting schemes, focusing on the possible effects of weighting criteria on the corresponding ratings. This extension requires more general conjunctions and implications.

We adopt the terminology and notations usual in multi-criteria decision making, where some alternatives are evaluated according to a common set $\mathcal{C} = \{1, \ldots, n\} = [n]$ of criteria. A common evaluation scale L is assumed to provide ratings according to the criteria: each alternative is thus identified with a function $f \in L^{\mathcal{C}}$ which maps every criterion i of $\mathcal{C}$ to the local rating f_i of the alternative with regard to this criterion.

We assume that L is a finite totally ordered set with 1 and 0 as top and bottom, respectively (L may be a subset $\{0 = a_0 < a_1 < \cdots < a_k = 1\}$ of the real unit interval [0, 1] for instance). For any $a \in L$, we denote by $\mathbf{a}_{\mathcal{C}}$ the constant alternative whose ratings equal a for all criteria in $\mathcal{C}$. In addition, we assume that L is equipped with a unary order reversing involutive operation $a \mapsto 1 - a$, that we call *negation*. In the finite setting, there is a unique such negation operation.[1] We respectively

[1] Often called strong negation in the literature, see [31].

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