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On periods and equilibria of computational sequential systems

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In this paper, we show that sequential systems with (Boolean) maxterms and minterms as global evolution operators can present orbits of any period. Besides, we prove that periodic orbits with different periods greater than or equal to 2 can coexist. Nevertheless, when a sequential dynamical system has fixed points, we demonstrate that periodic orbits of other periods cannot appear. Finally, we provide conditions to obtain a fixed point theorem in this context. This work provides a relevant advance in the knowledge of the dynamics of such systems which constitute one of the most effective mathematical tools to model computational processes and other phenomena from other Sciences. Moreover, the ideas developed here could help to obtain similar results for other related systems.

1. Introduction

In the last decade, graph dynamical systems (GDS) have emerged as one of the most effective mathematical tools to model not only computational processes, but also several phenomena of other Sciences. For this reason, the development of the theory of such systems has become a compelling need which has attracted the interest of many groups in mathematics applied to computation, as evidenced by the amount of works that it has generated.

In fact, this mathematical tool means a generalization of other ones which appeared previously in the literature as cellular automata (CA) [35,36] or Boolean networks (BN) [23,24]. This gives an idea of the versatility of this new paradigm in applications to several branches of Sciences as Biology (see [16,23,24,31]), Ecology (see [17,20,21]), Psychology (see [1,19]), Mathematics (see [13,14,18,22]), Physics (see [15]) and Chemistry (see [25]), among others.

To model a real-world phenomena and its evolution by GDS, it must be possible to decompose the corresponding system into the so-called *entities*, which are the lowest levels of aggregation of the system, where each entity i has assigned a (numerical) value x_i in relation to its state. The state value of each entity i can be formalized by considering $x_i \in \{0, 1\}$ if

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its possible states are only activated or deactivated. Nevertheless, the state values could belong to a general Boolean algebra (see [5]) or to a more general finite (resp. infinite) domain, so obtaining *finite range GDS* (resp. *generalized GDS*) (see [12]).

When the states of the entities are updated in a synchronous manner, the system is called a *parallel dynamical system* (PDS) (or alternatively *synchronous dynamical systems* (SyDS)) [2–5,11], while if they are updated in an asynchronous or sequential way, the system is named *sequential dynamical system* (SDS) [7–10,12,26,27]. In this last case, we also need a permutation which settles the order of updating of the state values of the entities.[1]

The relations among entities are represented by a graph called the *dependency graph* or *wiring diagram* of the system. This graph is usually assumed as connected; otherwise the study can be made for any connected component separately. Original CA and BN present restricted dependency graph in their definition (see [35] and [23]), while for GDS the dependency graph can be completely arbitrary. The graph can be undirected or directed in case of non-bidirectional relations between the entities. In this paper, unless specifically mentioned, the graph is considered undirected.

The evolution or update of the system is implemented by *local (evolution) functions* which together constitute a global (evolution) operator. When the local functions are restriction of a global one, we get an *homogenous* GDS or a GDS *induced* by this global function. However, the local functions can be considered completely independent as in [4,6,32].

Studying a dynamical system means to get to know its orbit structure [34], that is, to obtain as much information as possible about the phase space structure of such a system. As a first step in this study, it is usual to determine the length and number of coexisting periodic orbits. This is particularly interesting in this case, where the number of different states of the systems is finite, since every orbit is eventually periodic, i.e., finally arrives in a periodic one.

Concerning these questions, in [11] homogenous PDS and SDS are studied, considering the simplest Boolean functions OR (resp. AND) and NOR (resp. NAND) as global evolution operators. These results are extended in [3] for homogenous PDS with any general Boolean maxterm (resp. minterm) as global functions. In particular, it is proved that for such PDS uniquely periodic or eventually periodic orbits of period lower than or equal to 2 are possible. Furthermore, in the recent work [2], the authors have demonstrated that in such PDS the coexistence of orbits of different periods is impossible.

In this paper, we focus on the sequential counterpart, i.e., on homogenous SDS on Boolean maxterms (rep. minterms) as global updating operators. In this context, we find out that, in contrast with the case of PDS, periodic orbits of any period can appear for SDS, what supposes a breakthrough with respect to the results found for the parallel case. Moreover, we demonstrate that the coexistence of orbits of any different periods greater than or equal to two is possible, although we also prove that the existence of fixed points excludes the presence of periodic orbits of other periods. This shows that the existence of certain periods does not force the appearance of other ones in contrast with the Sharkovsky's order for discrete dynamical systems induced by continuous functions [29]. These results complete, to a certain extent, those on SDS in [8–10,26] or in Chapter 5 of [27] on phase-space structure of SDS.

An analogous study is made for non-homogenous PDS in [4], where the functions {*AND, OR, NAND, NOR*} are considered as local updating functions. More recently, in [33] a similar analysis of the limit cycle structure is performed for PDS and SDS governed by bi-threshold functions over non-uniform networks, which generalizes other works with static threshold values. In particular, they prove that for PDS on these other functions the maximal length of a periodic orbit is also 2.

Especially interesting is the study of fixed points, which can be divided in four more concrete problems to be solved, related to their existence, uniqueness and coexistence with other fixed points or periodic orbits.

- *Existence of fixed points* problem (EFP): Determining whether fixed points exist.
- *Coexistence of fixed points* problem (CFP): Determining whether fixed points are not the unique periodic orbits of a given system.
- *Unique fixed point* problem (UFP): Determining whether a given fixed point is the unique one.
- *Number of fixed points* problem (#FP): Counting the number of fixed points of a given system, in case of non-uniqueness.

In the literature, there are several attempts to solve each of these problems in different contexts. In [11], these problems are solved for homogenous PDS induced by the (global) Boolean functions *AND, OR, NAND* and *NOR*. In particular, for *AND* and *OR* only two fixed points appear (independently of the dependency graph of the system) which do not coexist with other periodic orbits; while for *NAND* and *NOR* no fixed point exists, appearing only 2-periodic orbits. In our recent paper [2], we solve the first three problems in the context of homogenous PDS on Boolean maxterms (minterms) as global evolution operators. Regarding the last problem #FP in this same context, some recent related works prove that it is computationally intractable (see [6,32]). Actually, only upper bounds for the number of fixed points are provided, based on the number of maximal independent sets of the dependency graph, even when the system is non-homogenous.

In the sequential case, some results regarding such problems can be also found in the literature. In Proposition 2 of [8], the authors demonstrate that fixed points of a SDS induced by symmetric Boolean functions as *NOR, NAND, PAR, MAJ, MIN* and *XOR* are independent of the predetermined order of updating. This result allows them to simplify the study to the (symmetric) parallel case. In this sense, in [9], they solve the aforementioned problems for the particular cases of parallel and sequential cellular automata (denoted by pCA and sCA respectively) which are equivalent to the study of SDS over a Circle graph $Circ_n$. They also generalize these results for SDS over $C_{n,\,r}$ graph [10], where any vertex is adjacent to the r

[1] The abbreviations GDS, PDS, SyDS, SDS, CA and BN will be written for the singular and plural forms of the corresponding terms, since it seems better from an aesthetic point of view.