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Value-at-risk in uncertain random risk analysis

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ABSTRACT

Uncertain random variables provide a tool to deal with phenomena in which uncertainty and randomness simultaneously exist. This paper proposes a concept of value-at-risk to quantify the risk of an uncertain random system. In addition, a value-at-risk theorem is proved in order to calculate the value-at-risk, and is applied to series systems, parallel system, *k*-out-of-*n* system, standby system, and structural system.

1. Introduction

In real life we constantly have to make decisions in random environment. To deal with this problem, probabilistic risk analysis was presented by Roy [18] with his risk index (safety-first criterion) which is the probability measure that some specified loss occurs. After that, probabilistic value-at-risk (VaR) was introduced by the leading bank J.P. Morgan [14] as a methodology to evaluate the maximum possible loss when the right tail distribution is ignored. In order to account for not only the probability of loss but also the severity of the loss, Rockafeller and Uryasev [19] presented the probabilistic tail value-at-risk (TVaR) that is the expected outcome conditional on the loss exceeding the VaR of the distribution. Probabilistic risk analysis has been successfully applied in engineering, finance, management science and so on.

In order to obtain the probability distribution function, adequate historical data are required. However, in many cases, there are no samples available to estimate the probability distribution. Thus some domain experts are invited to evaluate the degree of belief that each event may happen. On the one hand, human beings usually overweight unlikely events (Kahneman and Tversky [2]). On the other hand, human beings usually estimate a much wider range of values than the object actually takes (Liu [10]). This conservatism of human beings makes the degree of belief deviate far from the frequency. In order to rationally deal with degree of belief, uncertainty theory was founded by Liu [4] in 2007, and studied by many scholars subsequently. Based on uncertainty theory, uncertain programming was proposed by Liu [6] and was applied to machine scheduling problem, vehicle routing problem and project scheduling problem by Liu [10]. In addition, Liu [7] used uncertainty theory to evaluate the reliability index for uncertain systems. As an initial research on risk analysis in uncertain environment, Liu [7] defined risk as the accidental loss plus uncertain measure of such loss, and presented a risk index that is the uncertain measure that some specific loss occurs. In this way, Liu [7] built a framework of uncertain risk analysis based on his uncertainty theory. In addition, Peng [16] developed an uncertain value-at-risk (VaR) methodology and extended it to a tail value-at-risk (TVaR) methodology.

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In some cases, the uncertainty and randomness may coexist in a system. In order to deal with these phenomena, chance theory was pioneered by Liu [11] with the concepts of uncertain random variable and chance measure. Several papers deal with decision problems by using chance theory. Liu [12] proposed uncertain random programming with applications. Liu [9] introduced uncertain random graphs and uncertain random networks. Zhou et al. [21] introduced multi-objective optimization in uncertain random environments. Gao and Yao [1] introduced some concepts and results of uncertain random processes, and Ke et al. [3] introduced uncertain random multilevel programming with application to product control problem. Yao and Gao [20] introduced uncertain random alternating renewal processes with applications to interval availability. Recently, Qin [17] introduced uncertain random goal programming. Liu and Ralescu [13] defined risk index in uncertain random risk analysis, and a risk index theorem was proved in order to calculate the risk index. They also applied the uncertain random risk analysis to series systems, parallel systems, and standby systems.

This paper will introduce a tool of value-at-risk (VaR) to quantify the risk of an uncertain random system. Then a series of results will be proved in order to calculate the VaR. We will also apply the uncertain random VaR analysis to series systems, parallel systems, k-out-of-n systems, standby systems, and structural system.

2. Value-at-risk

Risk index (Liu and Ralescu [13]) is an important tool to deal with uncertain random system. Sometimes, we need to know how large is the scale of loss once the uncertain random loss happens with some degree. Taking this point of view, we introduce the following VaR metric of loss function, which is intuitively a combination of the chance measure of loss and the scale of loss.

Definition 1. Assume that a system contains uncertain random factors $\xi_1, \xi_2, \cdots, \xi_n$, and has a loss function f. Then the value-at-risk (VaR) is

$$\mathrm{VaR}(\alpha) = \sup\{x \,|\, \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq x\} \geq \alpha\}$$

for each given confidence level $\alpha \in (0,1]$.

When the uncertain random variables degenerate to random variables, the value-at-risk becomes the one in J.P. Morgan [14]. When the uncertain random variables degenerate to uncertain variables, the value-at-risk becomes the one in Peng [16].

Theorem 1. *The value-at-risk VaR(α) is decreasing with respect to α. That is, if $\alpha_1 < \alpha_2$, then* $\mathrm{VaR}(\alpha_1) \geq \mathrm{VaR}(\alpha_2)$.

Proof. It is easy to see from the definition of VaR that, if $\alpha_1 < \alpha_2$, then

$$\begin{aligned}\mathrm{VaR}(\alpha_1) &= \sup\{x \,|\, \mathrm{Ch}\{f(\xi_1, \cdots, \xi_n) \geq x\} \geq \alpha_1\} \\ &\geq \sup\{x \,|\, \mathrm{Ch}\{f(\xi_1, \cdots, \xi_n) \geq x\} \geq \alpha_2\} = \mathrm{VaR}(\alpha_2).\end{aligned}$$

Thus $\mathrm{VaR}(\alpha)$ is decreasing with respect to α. □

Theorem 2. *Assume that a system contains uncertain random factors $\xi_1, \xi_2, \cdots, \xi_n$, and has a loss function f. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ has a continuous chance distribution $\Phi(x)$, then for each confidence level $\alpha \in (0,1]$, we have*

$$\mathrm{VaR}(\alpha) = \sup\{x \,|\, \Phi(x) \leq 1 - \alpha\}.$$

Proof. It follows from the chance inversion theorem, duality of chance measure and continuity of chance distribution $\Phi(x)$ that

$$\begin{aligned}VaR(\alpha) &= \sup\{x \,|\, \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq x\} \geq \alpha\} \\ &= \sup\{x \,|\, 1 - \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) < x\} \geq \alpha\} \\ &= \sup\{x \,|\, 1 - \Phi(x) \geq \alpha\} \\ &= \sup\{x \,|\, \Phi(x) \leq 1 - \alpha\}.\end{aligned}$$

□

Theorem 3. *Assume that a system contains uncertain random factors $\xi_1, \xi_2, \cdots, \xi_n$, and has a loss function f. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ has a regular chance distribution $\Phi(x)$, then for each confidence level $\alpha \in (0,1]$, we have*

$$\mathrm{VaR}(\alpha) = \Phi^{-1}(1 - \alpha).$$