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Results on equality algebras

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ABSTRACT

In this paper, by considering the notion of equality algebra, which is introduced by Jenei in [18], as a possible algebraic semantic for fuzzy type theory, we study and show that there are relations among equality algebras and some of other logical algebras such as residuated lattice, *MTL*-algebra, *BL*-algebra, *MV*-algebra, Hertz-algebra, Heyting-algebra, Boolean-algebra, *EQ*-algebra and hoop-algebra. The aim of this paper is to find that under which conditions, equality algebras are equivalent to these logical algebras.

1. Introduction

Fuzzy type theory (FTT) [19,20] was developed as a counterpart of classical higher-order logic. Since the algebra of truth values is no longer a residuated lattice, a specific algebra called an *EQ-algebra* [21] for fuzzy type theory was proposed by Novák and De Baets. Hence, *EQ*-algebras are a generalization of residuated lattices. As it was mentioned in [18], if the product operation in *EQ*-algebras is replaced by another binary operation smaller or equal than the original product we still obtain an *EQ*-algebra, and this fact might make it difficult to obtain certain algebraic results. For this reason, a new structure was introduced by Jenei in [18], called *equality algebra* consisting of two binary operations meet and equivalence, and constant 1. It was proved in [10,12,18] that any equality algebra has a corresponding *BCK*-meet-semilattice and any *BCK*(*D*)-meet-semilattice (with distributivity property) has a corresponding equality algebra. Since equality algebras could also be candidates for a possible algebraic semantics for fuzzy type theory, their study is highly motivated. Equality algebras as well as pseudo equality algebras seem to have important algebraic properties and it is worth studying them from an algebraic point of view. In this paper, we study the relation among equality algebras and other logical algebras such as residuated lattices, *MTL*-algebra, *BL*-algebra, *MV*-algebra, Boolean-algebra, *EQ*-algebra and hoop-algebra [1,4,5,17,21–23].

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2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper and we shall not cite them every time they are used.

Definition 2.1. [18] An algebra $\mathcal{E} = (E, \wedge, \sim, 1)$ of the type (2, 2, 0) is called an *equality algebra*, if it satisfies the following conditions, for all $x, y, z \in E$:

(E1) $(E, \wedge, 1)$ is a commutative idempotent integral monoid (i.e. meet semilattice with top element 1),
(E2) $x \sim y = y \sim x$,
(E3) $x \sim x = 1$,
(E4) $x \sim 1 = x$,
(E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,
(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$. where $x \leq y$ if and only if $x \wedge y = x$, for all $x, y \in E$.

The operation $\wedge$ is called *meet (infimum)* and $\sim$ is an *equality operation*. Also, other two operations are defined, called *implication* and *equivalence operation*, respectively:

$$x \to y = x \sim (x \wedge y) \tag{I}$$

$$x \leftrightarrow y = (x \to y) \wedge (y \to x) \tag{II}$$

Definition 2.2. [17] A *BCK-algebra* is a structure $(B, \to, 1)$ of the type (2, 0) which satisfies the following conditions, for any $x, y, z \in B$:

(BCK1) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(BCK2) $1 \to x = x$,
(BCK3) $x \to 1 = 1$,
(BCK4) $x \to y = 1$ and $y \to x = 1$ imply $x = y$.

We can define a partial order $\leq$ on *BCK*-algebra B by $x \leq y$ if and only if $x \to y = 1$, for any $x, y \in B$. The equivalence operation $\leftrightarrow$ of B is defined by $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. A BCK-algebra B is called *bounded* if there exists an element $0 \in B$ such that $0 \leq x$, for all $x \in B$. If the poset $(B, \leq)$ is a meet-semilattice, then B is called a *BCK-meet-semilattice* and we denote it by $(B, \wedge, \to, 1)$, where $x \leq y$ if and only if $x \wedge y = x$. A BCK-meet-semilattice $(B, \wedge, \to, 1)$ is called a BCK-meet-semilattice with the *product* condition (*BCK(P)-meet-semilattice*), if it satisfies in the condition (P), that is for all $x, y \in B$, $x \odot y = \min\{z \mid x \leq y \to z\}$ exists. A BCK-meet-semilattice $(B, \wedge, \to, 1)$ is called a BCK-meet-semilattice with the *distributivity* condition (*BCK(D)-meet-semilattice*), if, for any $x, y \in B$, it satisfies in the condition (D): $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Proposition 2.3. *[10] Every BCK(P)-meet-semilattice is a BCK(D)-meet-semilattice.*

Theorem 2.4. *[10,12,18]The following two statements hold:*

(i) *For any equality algebra $\mathcal{E} = (E, \wedge, \sim, 1)$, $\Psi(\mathcal{E}) = (E, \wedge, \to, 1)$ is a BCK-meet-semilattice, where $\to$ denotes the implication of E.*
(ii) *For any BCK(D)-meet-semilattice $\mathcal{B} = (E, \wedge, \to, 1)$, $\Phi(\mathcal{B}) = (E, \wedge, \leftrightarrow, 1)$ is an equality algebra, where $\leftrightarrow$ denotes the equivalence operation of $\mathcal{B}$. Moreover, the implication of $\Phi(\mathcal{B})$ coincides with $\to$, that is, we have, $x \to y = x \leftrightarrow (x \wedge y)$.*

Definition 2.5 ([18]). Equality algebra $(E, \wedge, \sim, 1)$ (and as well as its equality operation $\sim$) called equivalential, if $\sim$ coincides with the equivalence operation of a suitably chosen equality algebra.

Theorem 2.6. *[18] An equality algebra $(E, \wedge, \sim, 1)$ is equivalential if and only if for all $x, y \in E$, $x \sim y = (x \sim (x \wedge y)) \wedge (y \sim (x \wedge y))$.*

By Theorem 2.6, (I) and (II), an equality algebra $(E, \wedge, \sim, 1)$ is equivalential if and only if for all $x, y \in E$, $x \sim y = x \leftrightarrow y$.
The following proposition provides some properties of equality algebras.

Proposition 2.7. *[18] Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be an equality algebra. Then the following properties hold, for all $x, y, z \in E$:*

(i) $x \sim y \leq x \leftrightarrow y \leq x \to y$,
(ii) $x \to y = 1$ *if and only if* $x \leq y$,
(iii) $1 \to x = x$, $x \to 1 = 1$, $x \to x = 1$,
(iv) $x \leq y \to x$,
(v) $x \leq (x \to y) \to y$,
(vi) $x \to y \leq (y \to z) \to (x \to z)$,
(vii) $x \leq y \to z$ *if and only if* $y \leq x \to z$,
(viii) $x \to (y \to z) = y \to (x \to z)$,
(ix) $y \leq x$ *implies* $x \leftrightarrow y = x \to y = x \sim y$.

Proposition 2.8. *[12] Let E be an equality algebra. Then, for any $x, y \in E$, if $x \leq y$, then $x \leq x \sim y$.*