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A Shapley measure of power in hierarchies

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ABSTRACT

A method for measuring positional power in hierarchies is proposed. Inspired by models of cooperative TU-games with restricted cooperation, such as permission structures, we model hierarchies by means of a certain kind of set games, which we have called authorization operators. We then define and characterize a value for authorization operators that allows us to quantify the power of each agent. This power is decomposed into two terms: sovereignty and influence. Sovereignty describes the autonomy of an agent. Influence indicates their capacity to block the actions of others.

1. Introduction

Numerous studies have been carried out on positional power in relational structures. Several of these studies are based on cooperative game theory. For instance, Molinero et al. [9] used the Shapley–Shubik and Banzhaf indices to measure power in social networks. In this paper, concepts of cooperative game theory will be employed to define a measure of power in hierarchies. Our starting point is the model for games with restricted cooperation introduced by Gilles et al. [6]. They introduced the concept of permission structure, which is a mapping that assigns to each agent a subset of direct subordinates. We aim to quantify the positional power of each agent in a permission structure. Since a permission structure is given by a digraph, the use of a measure of power in digraphs, such as that introduced by Herings et al. [7], could be contemplated. However, this measure would be unsuitable because, depending on the interpretation of the superior-subordinate relationship, a permission structure can represent different hierarchies. For instance, in the conjunctive approach, introduced by Gilles et al. [6] and generalized by Gallardo et al. [5], it is assumed that all agents need permission from all their superiors, whereas in the disjunctive approach, van den Brink [2] assumes that all agents need permission from just one of their predecessors. Other interpretations could also be considered. For instance, it could be assumed that all agents need permission from the majority of their predecessors. It can therefore be concluded that the same digraph can represent different hierarchical structures. That is, a digraph does not fully describe a hierarchy. It was hence decided to model hierarchies by means of another mathematical structure. To this end, we took into consideration that Gilles, Owen and van den Brink defined a set function that assigns to each coalition the set of players that are allowed to cooperate within that coalition. In fact, this set function fully describes the hierarchical structure. Bearing this in mind, we decided to model hierarchies by using a certain kind of set functions, called authorization operators, which are the crisp version of the fuzzy authorization operators introduced by Gallardo et al. [4]. Our next goal was to define a value for authorization operators. This value should

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hold information on the positional power in the structure. Since set functions are being dealt with here, the theory of set games introduced by Hoede [8] could be applied. A set game assigns to each coalition a subset of a set $\mathcal{U}$. In our case, $\mathcal{U}$ is the same set of agents. A value for set games assigns to each player a subset of $\mathcal{U}$. Aarts et al. [1] introduced a value for set games, which, within our framework, assigns, to each agent, the set of agents that depend on this agent. However, this value is not sensitive enough to distinguish how strongly an agent depends on another agent. In order to quantify positional power, it was therefore necessary to take a different approach. This led us to define a value for authorization structures as a correspondence that assigns, to each authorization structure, a mapping from the set of agents to the set of fuzzy coalitions. In this paper, a particular value for authorization structures is defined. It is shown that this value can be used to analyze positional power in any hierarchical structure. Moreover, this value allows us to evaluate the dependency relationships in these structures. This is used to introduce the concepts of sovereignty and influence, which are the two components of positional power.

The paper is organized as follows. In Section 2, several basic definitions and results concerning cooperative games, permission structures and set games are recalled. In Section 3, authorization structures, the concept of value for authorization structures and a particular value called the Shapley authorization correspondence are introduced. The indices of sovereignty and influence are also defined. In Section 4, the Shapley authorization correspondence and the indices of sovereignty and influence are characterized. In Section 5, some examples are provided. Finally, in Section 6, we draw conclusions and consider certain ideas for future research.

2. Preliminaries

2.1. Cooperative TU-games

A *transferable utility cooperative game or TU-game*is a pair (N, v) where N is a set of cardinality $n \in \mathbb{N}$ and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The elements of N are called *players*, the subsets $E \subseteq N$ are called *coalitions*and $v(E)$ is the *worth* of E. For each coalition E, the worth of E can be interpreted as the maximal gain or minimal cost that players in this coalition can achieve by themselves. Often, a TU-game (N, v) is identified with the function v. The family of all the games with set of players N is denoted by $\mathcal{G}^N$. This set is a $(2^n - 1)$-dimensional real vector space. A game $v \in \mathcal{G}^N$ is said to be monotonic if $v(E) \leqslant v(F)$ for every $E \subseteq F \subseteq N$. And v is superadditive if $v(E) + v(F) \leqslant v(E \cup F)$ for every $E, F \subseteq N$ with $E \cap F = \emptyset$.

A payoff vector for a game on the set of players N is a vector $x \in \mathbb{R}^N$. A *value*on$\mathcal{G}^N$ is a function $\psi : \mathcal{G}^N \longrightarrow \mathbb{R}^N$ that assigns a payoff vector to each game. Numerous values have been defined for several families of games in the literature. The *Shapley value*, introduced in [10], is the most important of these values. The Shapley value $Sh(v) \in \mathbb{R}^N$ of a game $v \in \mathcal{G}^N$ is a weighted average of the marginal contributions of each player to the coalitions. It is defined as

$$Sh_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E (v(E) - v(E \setminus \{i\})), \quad \text{for all } i \in N, \tag{1}$$

where $p_E = \dfrac{(n - |E|)!(|E| - 1)!}{n!}$, for every $E \subseteq N$.

In this paper, cooperative games that only take the values 0 and 1 are often used: these are called {0, 1}-games. A {0, 1}-game $v \in \mathcal{G}^N$ is said to be a simple game if $v(N) = 1$ and satisfies monotonicity. Shapley et al. [11] considered the Shapley value restricted to simple games, which is called the Shapley–Shubik index. Dubey [3] proved that the Shapley–Shubik index restricted to superadditive simple games is characterized by the following properties:

ANONYMITY. If $v \in \mathcal{G}^N$ is a superadditive simple game, π is a permutation of N and $i \in N$, then $\phi_i(\pi v) = \phi_{\pi(i)}(v)$, where $\pi v(E) = v(\pi(E))$ for every $E \subseteq N$.

NULL PLAYER. If $v \in \mathcal{G}^N$ is a superadditive simple game and i is a null player in v (i.e., $v(E \cup \{i\}) = v(E)$ for all $E \subseteq N$), then $\phi_i(v) = 0$.

TRANSFER. If $v, w \in \mathcal{G}^N$ are superadditive simple games, then $\phi(v) + \phi(w) = \phi(v \vee w) + \phi(v \wedge w)$, where $(v \vee w)(E) = \max(v(E), w(E))$ and $(v \wedge w)(E) = \min(v(E), w(E))$ for every $E \subseteq N$.

EFFICIENCY. If $v \in \mathcal{G}^N$ is a superadditive simple game, then $\sum_{i \in N} \phi_i(v) = 1$.

2.2. Permission structures

A *permission structure* on a finite set of players N is a mapping S: $N \to 2^N$. Given $i \in N$, the players in $S(i)$ are called *successors* of i in S. The transitive closure of S is denoted by $\hat{S}$, that is, $j \in \hat{S}(i)$ if and only if there exists a sequence $\{i_p\}_{p=0}^q$ such that $i_0 = i$, $i_q = j$ and $i_p \in S(i_{p-1})$ for all $1 \leq p \leq q$. The set of *predecessors* of i in S is denoted by $P_S(i) = \{j \in N : i \in S(j)\}$. The set of *superiors* of i in S is $\hat{P}_S(i) = \{j \in N : i \in \hat{S}(j)\}$. The collection of all permission structures on N is denoted by $\mathcal{S}^N$. A permission structure on N can be identified with a directed graph whose vertex set is N and whose edge set is $\{(i, j) \in N^2 : j \in S(i)\}$.

A *game with permission structure over* N is a pair (v, S) where $v \in \mathcal{G}^N$ and $S \in \mathcal{S}^N$. Numerous assumptions can be made about how a permission structure affects the possibilities of cooperation in a TU-game. Gilles et al. [6] considered the *conjunctive approach*, in which it is assumed that all players need the permission from all their superiors (if they have any)