Information Sciences 372 (2016) 625–633

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Several inequalities for the pan-integral

Yuchen Zhao[a], Tingsu Yan[b], Yao Ouyang[b,*]

[a] Deloitte Consulting (Shanghai) Co. Ltd., Room 2704 Bund Center, 222 Yan An Road East, Shanghai 200002, China
[b] Faculty of Science, Huzhou University, Huzhou, Zhejiang 313000, China

ARTICLE INFO

Article history:
Received 24 April 2016
Revised 14 August 2016
Accepted 19 August 2016
Available online 21 August 2016

Keywords:
Pan-integral
Subadditive measure
Chebyshev inequality
Hölder inequality
Minkowski inequality
Comonotonicity

ABSTRACT

Several inequalities for the pan-integral are investigated. It is shown that the Chebyshev inequality holds for an arbitrary subadditive measure if and only if the integrands f, g are comonotone. Thus, we provide a new characterization for nonnegative comonotone functions. It is also shown that the Hölder and Minkowski inequalities for the pan-integral hold if the monotone measure μ is subadditive. Since the pan-integral coincides with the concave integral when μ is subadditive, our results can also be applied to the concave integral.

1. Introduction

Monotone measures and related integrals (the Sugeno and Choquet integrals [4,23] for example) have been turned out to be useful in information system and related fields [5,11]. On the other hand, inequalities for classical integral are useful tools in several theoretical and applied fields. For example, the Hölder and Minkowski inequalities are crucial in L^p space theory, and Jensen's inequality plays a role in information theory. So the investigation of inequalities based on various nonlinear integrals is an important topic and we witness a large number of literatures in this area [3,19–21].

Take for example the Chebyshev inequality of the Sugeno and Choquet integrals. Mesiar and Ouyang [14,16] proved that, if f, g are comonotone (i.e., $(f(x)-f(y))(g(x)-g(y)) \geq 0$ holds for any $x, y \in X$ [22]), then

$$(S)\int_X f \star g d\mu \geq \left((S)\int_X f d\mu\right) \star \left((S)\int_X g d\mu\right) \tag{1.1}$$

holds for a continuous binary operation $\star$ on $[0, \infty)$ with some additional moderate conditions, while Girotto and Holzer [6] proved that two nonnegative functions f, g are comonotone if and only if

$$\mu(X)(S)\int_X \frac{1}{\mu(X)} fg d\mu \geq \left((S)\int_X f d\mu\right)\left((S)\int_X g d\mu\right) \tag{1.2}$$

holds for any monotone measure μ with $\mu(X) < \infty$. On the other hand, Girotto and Holzer [7] proved the Chebyshev inequality for the Choquet integral, that is, if nonnegative functions f, g are comonotone, then

$$\mu(X)(C)\int_X fg d\mu \geq \left((C)\int_X f d\mu\right)\left((C)\int_X g d\mu\right). \tag{1.3}$$

* Corresponding author.
E-mail addresses: yuczhao@deloitte.com.cn (Y. Zhao), yts@zjhu.edu.cn (T. Yan), oyy@zjhu.edu.cn (Y. Ouyang).

http://dx.doi.org/10.1016/j.ins.2016.08.067
0020-0255/© 2016 Published by Elsevier Inc.

They also proved that if (1.3) holds for any monotone measure μ with $\mu(X) < \infty$ then the nonnegative functions *f*, *g* are comonotone(see [7]). Noticing that $(C)\int_X \chi_X d\mu = \mu(X)$, (1.3) can be rewritten as

$$\left((C)\int_X \chi_X d\mu\right)\left((C)\int_X fgd\mu\right) \geq \left((C)\int_X fd\mu\right)\left((C)\int_X gd\mu\right). \tag{1.4}$$

Obviously, (1.4) also holds for the Lebesgue integral. We will see that (1.4) remains true for the pan- and concave integrals under some additional restrictions, whereas (1.3) may fail.

The Hölder and Minkowski inequalities also received much attention. Under the assumption that the monotone measure is submodular(see next section for the submodularity of monotone measure), Wang [27] proved these two inequalities for the Choquet integral, while the inequalities for Sugeno integral were discussed in [1,28].

In the literature, there are some other important types of nonlinear integrals, including, among others, the pan-integral [26] and the concave integral [8,9,13]. So far, the investigations of inequalities for these two integrals are missing. In this paper, we will go forward along this line. Concretely, we will discuss the Chebyshev, Hölder and Minkowski inequalities for the pan-integral. Our main assumption is the subadditivity of the monotone measures. Since the pan-integral coincides with the concave integral for a subadditive measure, our results remain true for the concave integral.

The paper is structured as follows. In Section 2, we recall some basic knowledge about monotone measure and the concave and pan- integrals. In Section 3, we prove the Chebyshev inequality under the frame of pan-integral, our result gives a new characterization for comonotonicity of nonnegative measurable functions. We discuss the Hölder and Minkowski inequalities for pan-integral in Section 4, and then conclude the paper with some concluding remarks.

2. Preliminaries

Let $(X, \mathcal{F})$ be a measurable space, i.e., *X* is a nonempty set and $\mathcal{F}$ a σ-algebra over *X*. Each element *A* of $\mathcal{F}$ is called a measurable set. All the sets appeared in this paper are supposed to be measurable. A set function $\mu : \mathcal{F} \to [0, \infty]$ is called a monotone measure if it satisfies:

(i) $\mu(\emptyset) = 0$ and $\mu(X) > 0$;
(ii) μ is monotone, that is $\mu(A) \leq \mu(B)$ for any $A, B \in \mathcal{F}$ with $A \subset B$.

A measurable space $(X, \mathcal{F})$ equipped with a monotone measure μ is called a monotone measure space. A monotone measure μ is called *continuous from below* if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \nearrow A$; *supermodularity* (or *convexity*) if $\mu(A\cup B) + \mu(A\cap B) \geq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{F}$; *superadditivity* if $\mu(A\cup B) \geq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{F}$ with $A\cap B = \emptyset$; *submodularity* (or *concavity*) if $\mu(A\cup B) + \mu(A\cap B) \leq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{F}$; *subadditivity* if $\mu(A\cup B) \leq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{F}$ with $A\cap B = \emptyset$ (see [4], for example). Note that, for a monotone measure, the submodularity (supermodularity, resp.) implies the subadditivity(superadditivity, resp.), but the converse is not true.

A function *f* defined over a measurable space $(X, \mathcal{F})$ is called measurable if, for each $B \in \mathcal{B}(\mathbb{R})$(the Borel algebra over $\mathbb{R}$), the preimage $f^{-1}(B)$ is an element of $\mathcal{F}$. In this paper, all the functions concerned are supposed to be measurable and nonnegative.

Let $(X, \mathcal{F}, \mu)$ be a monotone measure space and *f*: $X \to [0, \infty)$ be a nonnegative measurable function. Then the Choquet integral of *f* with respect to μ can be given by (see [13], for example)

$$(C)\int fd\mu = \sup\left\{\sum_{i=1}^n \lambda_i\mu(A_i) : \sum_{i=1}^n \lambda_i\chi_{A_i} \leq f, \right.$$
$$\left. \{A_i\}_{i=1}^n \subset \mathcal{F} \text{ is decreasing}, \lambda_i \geq 0, n \in \mathbb{N}\right\}.$$

and $(C)\int_A fd\mu = (C)\int_X f\cdot\chi_A d\mu$. Note that

$$(C)\int_A fd\mu = \int_0^\infty \mu(A\cap F_\alpha)d\alpha,$$

where the right-hand side integral is the improper Riemann integral and $F_\alpha = \{x : f(x) \geq \alpha\}$ is the α-level set of *f*.

For $A \in \mathcal{F}$, a sequence of sets $\{A_i\}_{i=1}^n \subset \mathcal{F}$ is said to be a *partition* of *A* if $\bigcup_i^n A_i = A$ and $A_i \cap A_j = \emptyset, i \neq j$. Recall that the pan-integral(with respect to $+, \cdot$) of a nonnegative measurable function *f* can be defined via

$$\int^{pan} fd\mu = \sup\left\{\sum_{i=1}^n \lambda_i\mu(A_i) : \sum_{i=1}^n \lambda_i\chi_{A_i} \leq f, \right.$$
$$\left. \{A_i\}_{i=1}^n \subset \mathcal{F} \text{ is a partition of } X, \lambda_i \geq 0, n \in \mathbb{N}\right\}.$$

and $\int_A^{pan} fd\mu = \int_X^{pan} f\cdot\chi_A d\mu$.

In [26] Wang and Klir proved that the pan-integral is greater than or equal to the Choquet integral if μ is subadditive and, the pan-integral is less than or equal to the Choquet integral if μ is superadditive.