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Typology of axioms for a weighted modal logic ☆

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ABSTRACT

This paper introduces and studies extensions of modal logics by investigating the soundness of classical modal axioms in a weighted framework. It discusses the notion of relevant weight values, in a specific weighted Kripke semantics and exploits accessibility relation properties. Different generalisations of the classical axioms are constructed and, from these, a typology of weighted axioms is built, distinguishing between four types, depending on their relations to their classical counterparts and to the, possibly equivalent, frame conditions.

1. Introduction

Weighted extensions of modal logics aim at increasing their expressiveness by enriching the two classical modal operators, $\Box$ and $\Diamond$, with integer or real valued degrees. These extensions, described in more detail in Section 2, are usually based on infinitely many weighted modal operators $\Box_\alpha$ and $\Diamond_\alpha$, where α stands for the numerical weight. These extended modalities make it possible to introduce fine distinctions between the pieces of knowledge modeled in the formalism, which can then be used to infer nuanced new knowledge and thus, for example, allow reasoning on partial beliefs [1–3]. To do this, one needs to define adequate versions of the weighted axioms to express relevant partial belief manipulation rules, i.e. to define a weighted extension of KD45.

With this objective in mind, this paper studies weighted extensions of the classical modal axioms, seen as rules defining the combination of the modal operators $\Box$ and $\Diamond$, establishing relations between formulae in which they occur once, repeatedly or in combination. For instance the classical axiom (4), $\vdash \Box\varphi \to \Box\Box\varphi$, states that an implication holds between a single occurrence and repetitions of $\Box$. Similarly, axiom (D), $\vdash \Box\varphi \to \Diamond\varphi$, establishes a relation between the two modal operators.

This paper examines the transposition of these axioms to the case of a weighted modal logic, identifying rules for the combination of the weighted modal operators $\Box_\alpha$ and $\Diamond_\alpha$. Starting with candidate weighted axioms obtained by replacing each modality of a classical axiom with a weighted one, each with its own weight, the paper discusses how these weights depend on each other. This issue can be illustrated by axiom (D), whose associated weighted candidate takes the form $\vdash \Box_\alpha\varphi \to \Diamond_\beta\varphi$. The question is then to establish a relevant valuation for β depending on α, or vice versa.

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The paper proposes to address this task from a semantic point of view, interpreting the candidates in a particular weighted Kripke semantics: it first proposes a semantic interpretation for $\Box_\alpha$ and $\Diamond_\alpha$ based on a relative counting of accessible validating worlds which relaxes the conditions on the universal quantifier defining $\Box$ in Kripke's semantics. This semantics offers the advantage of being informative enough to serve as a basis for the definition of weighted axioms.

The applied approach identifies weight dependencies which hold either in any frame or under specific frame conditions. The paper also studies whether specific conditions are satisfied by the frames in which the obtained axioms hold. This should be considered as opening the way to the definition of a weighted correspondence theory. Note that the aim here is not to introduce an axiomatisation of the considered weighted modal logic semantics, but to study the transposition of classical axioms to the weighted case: the semantic approach provides a motivation and justification for the proposed weighted axioms and their weight values, they can then be used as candidates for partial belief manipulation rules, or any other form of modal, non-factual reasoning.

The paper then introduces a typology of weighted modal axioms which separates them in four types, depending on their relation to their classic, unweighted counterparts and their associated frame conditions: type I groups axioms which cannot be relaxed using the degrees of freedom offered by the proposed weights. Type II is made of weighted axioms that preserve the frame conditions of their usual versions. Types III and IV contain weighted axioms that require a modification of the conditions imposed on the frame, respectively when correspondence cannot be proved and when it can.

The obtained typology allows to introduce, for instance, a weighted extension of KD45: the four axiom types will allow to identify axioms that are required by compatibility to the classic case, e.g. with the fact that the associated frames should satisfy some specific properties as well as axioms that may be additionally considered, depending on the desired behaviour of the manipulation rules.

The paper is organised as follows: Section 2 presents an informal comparative study of existing weighted modal logics. Section 3 introduces the semantics used to build weighted axioms with the method described in Section 4. Section 5 presents an overview of the resulting typology of weighted modal axioms, whereas Sections 6 to 9 are dedicated to each type in turn.

2. Existing weighted modal logics

After presenting the notations used in this paper, this section briefly describes existing weighted extensions of modal logics, first with approaches which modify the definition of Kripke frames, integrating weights either in the accessibility relation or in the worlds. It then describes the counting models, which preserve the classical frame definition but alter the quantification used in the modal operator definitions.

Specific weighted modal systems, dedicated to particular applications, are not detailed in this section. These include, for instance, fuzzy temporal logic [4] or multi-agent modal logic [5].

2.1. Notations

We adopt the standard notation (e.g. see [6,7]): a frame $F = \langle W, R\rangle$ is a pair composed of a non-empty set W of worlds and a binary accessibility relation R on W. A model $\mathcal{M} = \langle F, s\rangle$ is a couple formed by a frame F and a valuation s which assigns truth values to each atomic formula, in each world in W.

For any formula φ and any world $w \in W$, the usual definition of the semantic consequence symbol $\models$ states that $\mathcal{M}, w \models \varphi$ if and only if φ is true in world w for the considered model $\mathcal{M}$ (the latter may be omitted when there is no ambiguity). The notation $F \models \varphi$ means that for any valuation s and for any world $w \in W$, $\langle F, s\rangle, w \models \varphi$. Finally, $\models \varphi$ means that for any frame $F = \langle W, R\rangle$, any valuation s and any world $w \in W$, $\langle F, s\rangle, w \models \varphi$.

For a given model $\mathcal{M}$ and any world w in W, we denote by R_w the set of worlds accessible from w:

$$R_w = \left\{w' \in W \mid wRw'\right\} \tag{1}$$

We also define, for any formula φ, the set $R_w(\varphi)$:

$$R_w(\varphi) = \left\{w' \in R_w \mid \mathcal{M}, w' \models \varphi\right\} \tag{2}$$

For any formula φ, the classical interpretations of $\Box\varphi$ and $\Diamond\varphi = \neg\Box\neg\varphi$ are respectively based on the universal and existential quantification of accessible worlds which satisfy φ. To use the previous notations, one can write:

$$\begin{aligned}\mathcal{M}, w \models \Box\varphi \quad &\Leftrightarrow \quad \forall w' \in R_w,\ \mathcal{M}, w' \models \varphi \qquad (3)\\ &\Leftrightarrow \quad R_w(\varphi) = R_w\\ \mathcal{M}, w \models \Diamond\varphi \quad &\Leftrightarrow \quad \exists w' \in R_w,\ \mathcal{M}, w' \models \varphi \qquad (4)\\ &\Leftrightarrow \quad R_w(\varphi) \neq \emptyset\\ &\Leftrightarrow \quad |R_w(\varphi)| > 0\end{aligned}$$