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On linearity of pan-integral and pan-integrable functions space

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A B S T R A C T

This paper investigates the linearity and integrability of the $(+,\cdot)$-based pan-integrals on subadditive monotone measure spaces. It is shown that all nonnegative pan-integrable functions form a convex cone and the restriction of the pan-integral to the convex cone is a positive homogeneous linear functional. We extend the pan-integral to the general real-valued measurable functions. The generalized pan-integrals are shown to be symmetric and fully homogeneous, and to remain additive for all pan-integrable functions. Thus for a subadditive monotone measure the generalized pan-integral is linear functional defined on the linear space which consists of all pan-integrable functions. We define a p-norm on the linear space consisting of all p-*th* order pan-integrable functions, and when the monotone measure μ is continuous we obtain a complete normed linear space $L^p_{pan}(X,\mu)$ equipped with the p-norm, i.e., an analogue of classical Lebesgue space L^p.
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1. Introduction

In nonlinear integral theory there are three types of important integrals, the Choquet integral [2], the pan-integral (based on the usual addition $+$ and multiplication $\cdot$ on reals) [37] and the concave integral [12,13]. These integrals coincide with the Lebesgue integral in all cases for σ-additive measures, i.e., these three nonlinear integrals (with respect to monotone measures) are particular generalizations of the Lebesgue integral. In general, they are significantly different from each other, the Choquet integral deals with finite chains of sets, the pan-integral is based on finite partitions (disjoint set systems, similar to the Lebesgue integral) while the concave integral is related to arbitrary finite set systems, see [21]. These integrals have been widely studied and many important results have been obtained, for more details see [3,4,8,15,27,28,33,35], etc., and the structure theory of integral has been enriched and developed in depth, see [7,10,11,17,18,21].

It is well-known that L^p space theory is a crucial aspect of classical measure theory [1,5,6]. Since the above mentioned three integrals are based on monotone measures and lack additivity, in general case the L^p space theory is not true for these integrals. Denneberg [3] showed a subadditivity theorem for the Choquet integral under the assumption that the monotone measure is submodular. The corresponding L^p space theory for the Choquet integral was developed. The similar results were presented by Shirali [30], and the related researches were presented in recent paper by Pap [29]. Note that the Choquet integral is a level set-based integral, which ensures its comonotonic additivity – a property weaker than additivity.

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In [19] the linearity and generalized linearity of fuzzy integral (including the Choquet integral and the Sugeno integral) were discussed (see also [9]). Unlike the Choquet integral, the pan-integral is a partition-based integral. The relationships between these two types of integrals were discussed in [16,25,26] (see also [35]). As a result, the pan-integral even lacks comonotonic additivity. From this point of view, it requires to examine whether or not the L^p space theory holds for the pan-integral. In this paper we will investigate this problem.

The concept of pan-integral introduced by Yang ([35,37]) involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers, i.e., it considers the commutative isotonic semiring $(\overline{R}_+, \oplus, \otimes)$ (see [14,27,32,35,36]). A related concept of generalized Lebesgue integral based on a generalized ring was proposed and discussed (see [39]). In this paper we only consider the pan-integrals based on the standard addition $+$ and multiplication $\cdot$.

This paper is structured as follows. After this introduction, in the next section we recall some basic facts about the monotone measures and the pan-integrals. In Sections 3 and 4 we consider the pan-integral for nonnegative measurable functions. We show that on a subadditive monotone measure space $(X, \mathcal{A}, \mu)$ the pan-integral with respect to μ is additive. Noting that the pan-integral based on the standard addition $+$ and multiplication $\cdot$ is positively homogeneous, then all nonnegative pan-integrable functions form a convex cone in the usual sense and thus the restriction of the pan-integral to the convex cone is a positive homogeneous linear functional. In Section 5, in the same way as the symmetric Choquet integral ([3,27]) we extend the pan-integral for nonnegative measurable functions to the class of general real-valued measurable functions (not necessarily nonnegative). We get a type of symmetric and fully homogeneous nonlinear integral. For a subadditive monotone measure μ the generalized pan-integral (with respect to μ) remains additive for all pan-integrable functions, and hence it is a linear functional defined on the linear space consisting of all pan-integrable functions. In Section 6, we show that all p-*th* order pan-integrable functions form a linear space under the conditions that the monotone measure is subadditive and continuous from below. On such the linear space, by using the Minkowski type inequality for pan-integral [38], we can define a p-norm in the usual manner as Lebesgue space L^p and then we obtain a complete normed linear space $L^p_{pan}(X, \mu)$ equipped with the p-norm, i.e., it is analogous to Lebesgue space L^p. Thus, in the framework of the generalized pan-integral the classical L^p space is generalized.

2. Preliminaries

Let X be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of X. A set function $\mu : \mathcal{A} \to [0, +\infty]$ is called a *monotone measure* [35] on $(X, \mathcal{A})$, if it satisfies the following conditions:

(1) $\mu(\emptyset) = 0$ and $\mu(X) > 0$;
(2) $\mu(A) \le \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{A}$.

When μ is a monotone measure, the triple $(X, \mathcal{A}, \mu)$ is called a monotone measure space ([27,35]).

Note: In the literature, the monotone measure is also known as a monotone set function, a capacity, a fuzzy measure, or a nonadditive probability (constrained by $\mu(X) = 1$, sometimes also assuming the continuity of μ), etc. (see [2,3,15,22,27,31,34,36]).

In this paper we always assume that μ is a monotone measure on $(X, \mathcal{A})$. Recall that μ is said to be

(i) *subadditive*, if $\mu(A \cup B) \le \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$;
(ii) *submodular*, if $\mu(A \cup B) + \mu(A \cap B) \le \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$;
(iii) *supermodular*, if $\mu(A \cup B) + \mu(A \cap B) \ge \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$;
(iv) *null-additive* [35], if for any $A, B \in \mathcal{A}$, $\mu(B) = 0$ implies $\mu(A \cup B) = \mu(A)$;
(v) *continuous from below*, if for any $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$, $A_1 \subset A_2 \subset \ldots$ implies $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;
(vi) *continuous from above*, if for any $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$, $A_1 \supset A_2 \supset \ldots$ and $\mu(A_1) < \infty$ imply $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;
(vii) *continuous* if it is continuous both from below and from above.

Obviously, the submodularity of μ implies the subadditivity and both imply the null-additivity, but not vice versa.

In [37] (see also [23,35,39]) the concept of pan-integral was introduced, which involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers (see [20,27,32,35]). In this paper we only consider the pan-integrals based on the standard addition $+$ and multiplication $\cdot$.

A real-valued function $f : X \to (-\infty, +\infty)$ is said to be $\mathcal{A}$-measurable on $(X, \mathcal{A})$ (or simply "measurable" when there is no confusion) if $f^{-}(B) \in \mathcal{A}$ for every Borel set B of real numbers. Let $\mathcal{F}^+$ be the collection of all nonnegative real-valued measurable functions on $(X, \mathcal{A})$. We recall the following definition.

Definition 2.1. Let $(X, \mathcal{A}, \mu)$ be a monotone measure space, $A \in \mathcal{A}$ and $f \in \mathcal{F}^+$. The pan-integral of f on A with respect to μ, is defined by

$$\int_A^{pan} f d\mu = \sup_{\mathcal{E} \in \hat{\mathcal{P}}} \left\{ \sum_{E \in \mathcal{E}} \left[\left(\inf_{x \in A \cap E} f(x) \right) \cdot \mu(A \cap E) \right] \right\}, \tag{2.1}$$

where $\hat{\mathcal{P}}$ is the set of all finite measurable partitions of X. When $A = X$, $\int_X^{pan} f d\mu$ is written as $\int^{pan} f d\mu$. If $\int_A^{pan} f d\mu < \infty$, then we say that f is pan-integrable on A. When f is pan-integrable on X, we simply say that f is pan-integrable.

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