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Copulas with given values on the tails

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ABSTRACT

We provide sufficient conditions for constructing bivariate copulas with given values in rectangles at the corners of the unit square, and find best-possible bounds for copulas of this type.

1. Introduction

The issue of aggregation of different sources of information into a single output has obtained a number of theoretical results in the recent years; however, the choice of the aggregation function still remains a crucial problem, especially in view of its impact in applications. In order to guide the choice of the aggregation procedures, several construction methods have been introduced assuming that the values of the aggregation are known in particular instances (see, e.g., [19] for an overview).

A copula is a special kind of n-ary aggregation function whose input values belong to $[0, 1]$. Copulas are also Lipschitz continuous aggregation functions, i.e. the error in the aggregation output is linearly dependent on the error in the corresponding input terms (see [12,20] for more details). Copulas have been extensively used for modelling uncertainty of different types, from probabilistic methods (see [21,27]) to imprecise probabilities and decision theory (see [24,26,35,36,40]). In such settings, various construction methods for copulas have been introduced assuming that the values of the copula are known in specific areas of the domain. See, for instance, [6,8,10,14,22,23] among others. Roughly speaking, these methods can help the decision maker in the choice of a suitable copula model and/or in providing optimal bounds for the behaviour of the multivariate system of interest. In financial applications, for instance, model-free bounds have been considered for the prices for two-dimensional portfolios when some information on the copula is available; see, e.g., [2,38].

Here, we are interested in the class of copulas that have specified values in the corners of the copula domain. Such corners are of particular relevance in risk management problems since they often determine the system behaviour in hazard

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situations (see, for instance, [33]). Our results are also connected to the problem of finding ways to extend a given subcopula to a copula, which has relevant implications in the use of copulas for modelling mixed-type data (see, for instance, [5,7,17,37]).

The paper is organized as follows. After some preliminaries concerning copulas and quasi-copulas (Section 2), we provide in Section 3 sufficient conditions for constructing copulas with given values on rectangles at the corners of the unit square, and find best-possible bounds for copulas in this class.

2. Preliminaries

Here, we recall some basic results about copulas and quasi-copulas that will be necessary in the sequel. For details, we refer to, e.g., [12,21,27].

Let $A \subseteq \mathbb{R}^2$. For a function $F\colon A \to [0, +\infty[$ and a rectangle $R = [x_1, x_2] \times [y_1, y_2] \subseteq A$, we denote by

$$V_F(R) := F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$$

the so-called F-*volume* of R.

A (bivariate) *copula* is a binary operation $C\colon [0,1]^2 \longrightarrow [0,1]$ which satisfies

(C1) the boundary conditions $C(t, 0) = C(0, t) = 0$ and $C(t, 1) = C(1, t) = t$ for every $t \in [0, 1]$;
(C2) the *2-increasing* property, i.e., $V_C(R) \geq 0$ for every rectangle $R \subseteq [0,1]^2$.

Since our methods involve quasi-copulas as well as copulas, we recall that a *quasi-copula* (see, e.g., [1,12,18,19]) is a function $Q : [0,1]^2 \longrightarrow [0,1]$ which satisfies condition (C1) of copulas, but in place of (C2), the weaker conditions

(Q1) Q is non-decreasing in each variable;
(Q2) the *Lipschitz* condition $|Q(u_1, v_1) - Q(u_2, v_2)| \leq |u_1 - u_2| + |v_1 - v_2|$ for all (u_1, v_1), (u_2, v_2) in $[0,1]^2$.

While every copula is a quasi-copula, there exist *proper* quasi-copulas, i.e., quasi-copulas which are not copulas — distinctions concerning the mass distribution of copulas and (proper) quasi-copulas can be found in [16,29].

One of the most important occurrences of quasi-copulas in statistics is due to the following observation [15,28,30]: Every set $\mathscr{S}$ of (quasi-)copulas has the smallest upper bound and the greatest lower bound in the set of quasi-copulas (in the sense of pointwisely ordered functions). These bounds do not necessarily belong to the set $\mathscr{S}$, nor they are necessarily copulas whenever $\mathscr{S}$ does not contain any quasi-copula. In particular, the best-possible bounds for the set of all quasi-copulas are given by the *Fréchet–Hoeffding* bounds, i.e., for any quasi-copula Q we have

$$W(u, v) := \max(0, u + v - 1) \leq Q(u, v) \leq \min(u, v) =: M(u, v) \tag{1}$$

for all $(u, v) \in [0,1]^2$; furthermore, the bounds W and M are themselves copulas.

Fréchet–Hoeffding bounds for (quasi-)copulas can be improved when we know additional inequality constraints on the specific class under consideration. For instance, best-possible bounds have been given in [38] for the set of quasi-copulas that coincide on a given compact subset S of $[0,1]^2$ (see also [2,3]). Here, we recall such a result that will be used in the following.

Theorem 2.1. *Let S be a compact subset of $[0,1]^2$ and let Q be a quasi-copula. Then, for every quasi-copula Q' that coincides with Q on S and for all $(u, v) \in [0,1]^2$, it holds*

$$B^{S,Q}(u, v) \leq Q'(u, v) \leq A^{S,Q}(u, v)$$

where

$$B^{S,Q}(u, v) = \max\left\{W(u, v), \max_{(x,y)\in S}\left\{Q(x, y) - (x - u)^+ - (y - v)^+\right\}\right\}$$

$$A^{S,Q}(u, v) = \min\left\{u, v, \min_{(x,y)\in S}\left\{Q(x, y) + (u - x)^+ + (v - y)^+\right\}\right\},$$

with $z^+ = \max\{z, 0\}$.

Best-possible bounds for a set of copulas when the value at a single point is known have been discussed in [27] (see also [34]). For a more detailed overview, see also [4,31,32].

Finally, we recall the notion of ordinal sum (see, e.g., [12,27]). Let $(J_i)_{i\in\mathscr{I}} = ([a_i, b_i])_{i\in\mathscr{I}}$, $a_i < b_i$, be a collection of subintervals of $[0,1]$ indexed by $\mathscr{I}$ such that $[a_i, b_i]$ at most overlap at the interval bounds, i.e., $[a_{i_1}, b_{i_1}] \cap [a_{i_2}, b_{i_2}]$ is either empty or a singleton for any $i_1 \neq i_2$. Let $(C_i)_{i\in\mathscr{I}}$ be a collection of (quasi-)copulas. Then, the *ordinal sum* of the collection $(C_i)_{i\in\mathscr{I}}$ with respect to $(J_i)_{i\in\mathscr{I}}$ is the (quasi-)copula C defined by