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The effect of combination functions on the complexity of relational Bayesian networks ☆

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ABSTRACT

We study the complexity of inference with Relational Bayesian Networks as parameterized by their probability formulas. We show that without combination functions, inference is PP-complete, displaying the same complexity as standard Bayesian networks (this is so even when the domain is succinctly specified in binary notation). Using only maximization as combination function, we obtain inferential complexity that ranges from PP-complete to PSPACE-complete to PEXP-complete. And by combining mean and threshold combination functions, we obtain complexity classes in all levels of the counting hierarchy. We also investigate the use of arbitrary combination functions and obtain that inference is EXP-complete even under a seemingly strong restriction. Finally, we examine the query complexity of Relational Bayesian Networks (i.e., when the relational model is fixed), and we obtain that inference is complete for PP.

1. Introduction

Bayesian networks provide an intuitive language for the probabilistic description of concrete domains [22]. Jaeger's Relational Bayesian Networks, here referred to as RBNs, extend Bayesian networks to abstract domains, and allow for the description of relational, context-specific, deterministic and temporal knowledge [20,21]. There are many languages that also extend Bayesian networks into relational representations [11,12,16,23,24,30]; RBNs offer a particularly general and solid formalism.

RBNs constitute a specification language containing a small number of constructs: relations, probability formulas, combination functions, and equality constraints. Combination functions are a particularly important modeling feature, as they provide a way of aggregating information from different elements of the domain.

It should not be surprising that the inferential complexity of Bayesian networks specified by RBNs depends on the choice of constructs allowed. However, few results have been produced on the relation between the expressivity of such constructs and the complexity of inference.

In this paper, we examine the effect of combination functions on the complexity of inferences with (Bayesian networks specified by) RBNs. We first argue that, without combination functions, RBNs simply offer a language that is similar to *plate models*, a well-known formalism to describe models with simple repetitive structure [17,27]. We show that without

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Table 1
Summary of complexity of inference parameterized by the combination functions allowed, how domain is specified, the maximum arity of relations, and the maximum nesting level of combination expressions.

Combination functions	Domain spec.	Bounded arity?	Bounded nesting?	Complexity of inference
none	unary	yes	–	PP-complete
none	binary	yes	–	PP-complete
max	binary	yes	yes	PEXP-complete
max	unary	no	yes	PEXP-complete
max	unary	yes	no	PSPACE-complete
max	unary	yes	yes	$\mathrm{PP}^{\Sigma_k^P}$-complete
threshold, mean	unary	yes	yes	$\mathrm{C}_k\mathrm{P}$-hard[a]
polynomial	unary	yes	yes	EXP-complete

[a] Membership when threshold and mean are used depend on further constrains explained in Section 4.3.

combination functions inference is PP-complete, irrespective of the encoding of the domain; this matches the complexity of inference in standard (propositional) Bayesian networks. When we allow combination functions and the associated equality constraints into the language, matters complicate considerably. When either the domain is specified in binary or the arity of relations is unbounded, inference is PEXP-complete even when the only combination function is maximization. If we place a bound on arity, and specify the domain in unary notation (or equivalent, as an explicit list of elements), and allow only maximization as combination function, then inference is PSPACE-complete. This is mostly generated by the ability to nest an unbounded number of maximization expressions. In fact, by further restricting the number of nesting of combination functions, we obtain the same power as a probabilistic Turing machine with access to an oracle in the polynomial hierarchy. We then look at a combination of mean and a threshold: the former allows probabilities to be defined as proportions; the latter allows the specification of piecewise functions. We argue that threshold and mean combined are as powerful as maximization, and thus all previous results hold. And by a suitable constraint on the use of threshold and mean, we show that we can obtain complexity in every class of the counting hierarchy. We also look at the complexity of inference when the combination function is given as part of the input. The challenge here is to constrain the language so as to obtain non-trivial complexity results. We show that requiring polynomial-time combination functions is too weak a condition in that it leads to EXP-complete inference. On the other hand, requiring polynomially long probability formulas brings inference down to PP-completeness. These results are summarized in Table 1.

We also investigate the complexity of inference when the RBN is assumed fixed. This is equivalent to the idea of compiling a probabilistic model [6,9]. We show that complexity is either polynomial if probability formulas can be computed in polynomial time (which includes the case of no combination expressions) or PP-complete, when the combination functions can be computed in polynomial time (which includes the cases of maximization, threshold and mean).

The paper begins with a brief review of RBNs (Section 2), and key concepts from complexity theory (Section 3). Our contributions regarding inferential complexity appear in Section 4. The complexity of inference without combination functions appear in Section 4.1. Relational Bayesian networks allowing only combination by maximization are analyzed in Section 4.2, while networks allowing mean and threshold are analyzed in Section 4.3. General polynomial-time computable combination formulas are examined in Section 4.4. Query complexity is discussed in Section 5. We justify our use of decision problems (instead of functional problem) and discuss how our results can be adapted to provide completeness for classes in the functional counting hierarchy in Section 6. A summary of our contributions and open questions are presented in Section 7.

2. Relational Bayesian networks

2.1. Bayesian networks

A Bayesian network is a compact description of a probabilistic model over a propositional language [7,22]. It consists of two parts: an acyclic directed graph $G = (V, A)$ over a finite set of random variables $X_1, \dots, X_n$, and a set of conditional probability distributions, one for each variable and each configuration of its parents. The parents of a variable X in V are denoted by $\mathrm{pa}(X)$. In this paper, we consider only 0/1-valued random variables, hence each conditional distribution $\mathbb{P}(X|\mathrm{pa}(X))$ can be represented as a table.

The semantics of a Bayesian network is obtained by the directed Markov property, which states that every variable is conditionally independent of its non-descendants given its parents. For categorical random variables, this assumption induces a single joint probability distribution by

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i|\mathrm{pa}(X_i) = \pi_i)\,,$$

where π_i is the vector of values for $\mathrm{pa}(X_i)$ induced by assignments $\{X_1 = x_1, \dots, X_n = x_n\}$. Bayesian networks can represent complex propositional domains, but lack the ability to represent relational knowledge.