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Probabilistic squares and hexagons of opposition under coherence ☆,☆☆

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ABSTRACT

Various semantics for studying the square of opposition and the hexagon of opposition have been proposed recently. We interpret *sentences* by imprecise (set-valued) probability assessments on a finite sequence of conditional events. We introduce the *acceptability* of a sentence within coherence-based probability theory. We analyze the relations of the square and of the hexagon in terms of acceptability. Then, we show how to construct probabilistic versions of the square and of the hexagon of opposition by forming suitable tripartitions of the set of all coherent assessments on a finite sequence of conditional events. Finally, as an application, we present new versions of the square and of the hexagon involving generalized quantifiers.

1. Introduction

There is a long history of investigations on the square of opposition spanning over two millennia [5,49]. A *square of opposition* represents logical key relations among basic (syllogistic) sentence types in a diagrammatic way. The basic sentence types, traditionally denoted by A (universal affirmative: "Every S is P"), E (universal negative: "No S is P"), I (particular affirmative: "Some S is P"), and O (particular negative: "Some S is not P"), constitute the corners of the square. The diagonals and the sides of the square of opposition are formed by the following logical relations among the basic sentence types: A and E are *contraries* (i.e., they cannot both be true), I and O are *subcontraries* (i.e., they cannot both be false), A and O as well as E and I are *contradictories* (i.e., they cannot both be true and they cannot both be false), I is a *subaltern* of A and O is a *subaltern* of E (i.e., A entails I and E entails O); for a visual representation see Fig. 1 below, and cover the probabilities for seeing the traditional square of opposition). In the early 1950ies, the square of opposition was expanded to the *hexagon of opposition*, by adding the sentence $U : A \vee E$ at the top and the sentence $Y : I \wedge O$ at the bottom of the square (see Fig. 2). Recently, the square of opposition as well as the hexagon of opposition and its extensions have been investigated from various semantic points of view (see, e.g., [4,5,14,24–26,33,45–47]). In this paper we present a probabilistic analysis of

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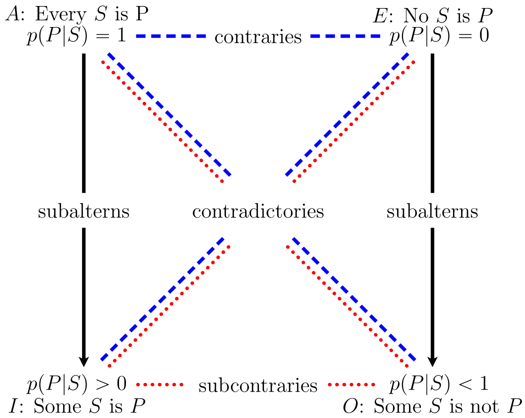

Fig. 1. Traditional and probabilistic square of opposition defined on the four classical sentence types A, E, I, O and their relations in between. The probabilistic semantics of the basic sentence types involving the predicate term P and the subject term S is interpreted by a suitable probability assessment on the conditional event $P|S$ (see Table 1). For the relations see Definition 9.

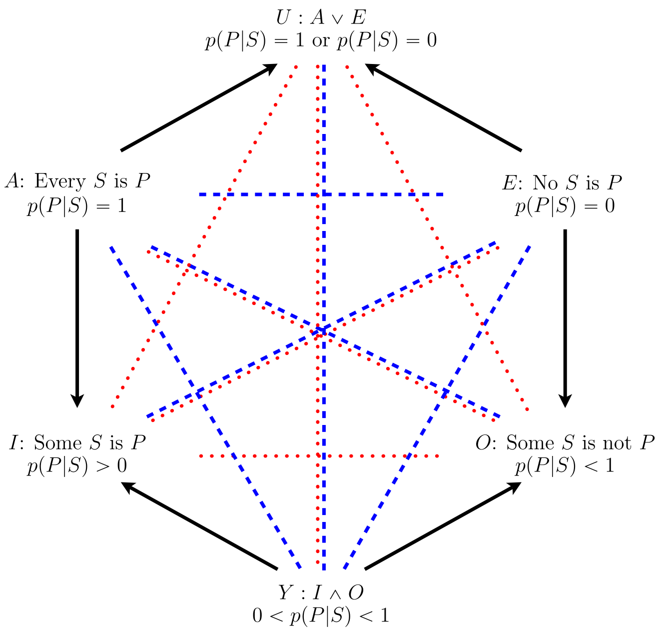

Fig. 2. Probabilistic hexagon of opposition on the six sentence types A, E, I, O, U, Y, where A, E, I, O is a square of opposition, $U = A \vee E$ and $Y = I \wedge O$. The arrows indicate subalternation, dashed lines indicate contraries, and dotted lines indicate sub-contraries. Contradictories are indicated by combined dotted and dashed lines.

the square of opposition under coherence, introduce the hexagon of opposition under coherence, and study the semantics of basic key relations among quantified statements.

After preliminary notions (Section 2), we introduce, based on g-coherence, a (probabilistic) notion of sentences and their acceptability and show how to construct squares of opposition under coherence from suitable tripartitions (Section 3). Then, we present an application of our square to the study of generalized quantifiers (Section 4). In Section 5 we introduce the *hexagon of opposition* under coherence. Section 6 concludes the paper by some remarks on future work.

2. Preliminary notions

The coherence-based approach to probability and to other uncertain measures has been adopted by many authors (see, e.g., [6,8,12,13,17–22,29,31,36–38,41,50,51,55,56,58,63]); we therefore recall only selected key features of coherence and its generalizations in this section.

An event E is a two-valued logical entity which can be either true or false. The indicator of E is a two-valued numerical quantity which is 1, or 0, according to whether the event E is true, or false, respectively. We use the same symbols for events and their indicators. We denote by $\top$ the sure event (i.e., tautology or logical truth) and by $\bot$ the impossible event