ARTICLE IN PRESS
JID:IJA AID:8077 /FLA [m3G; v1.218; Prn:30/06/2017; 11:58] P.1 (1-14)
International Journal of Approximate Reasoning ••• (••••) •••–•••

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Basic ideas underlying conglomerability and disintegrability ☆

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A R T I C L E I N F O

Article history:
Received 20 December 2016
Received in revised form 15 June 2017
Accepted 16 June 2017
Available online xxxx

Keywords:
Coherence
Conglomerability
Disintegrability
Finitely additive probability
Lower prevision
Natural extension

A B S T R A C T

The basic mathematical theory underlying the notions of conglomerability and disintegrability is reviewed. Both the precise and the imprecise cases are concerned.

1. Introduction

In time, conglomerability and disintegrability have been investigated by various authors, in different frameworks and under different assumptions. This produced a slightly chaotic situation with risk of misunderstandings. As a result, conglomerability/disintegrability may look much more involved than they are.

For instance, it is not always clear whether or not conglomerability implies disintegrability; see e.g. [1, page 206]. Instead, this issue is quite simple: It depends on the class of random variables where the various probability evaluations are defined. Or else, sometimes, one speaks about "conglomerability according to X" and "conglomerability according to Y", where X and Y are different authors, raising the doubt of the existence of more than one notion of conglomerability. There is only one notion, instead, which is merely applied in different settings. (The situation is more involved in the so called "imprecise case", introduced below.)

This paper does not include new results but aims to bring out and make precise the basic ideas underlying conglomerability and disintegrability. We focus on the general (mathematical) theory more than on results concerning specific problems.

So far, we referred to the so called *precise* case. Roughly speaking, "precise" means that probability evaluations are (at least) finitely additive. Recently, however, there is a growing interest on the *imprecise* case, where finite additivity is not required and weaker types of probability evaluations come into play. Accordingly, in the second part of this paper, conglomerability/disintegrability are discussed in the imprecise case. There is however a notable difference with the precise

☆ This paper is part of the Virtual special issue on Soft methods in probability and statistics, Edited by Barbara Vantaggi, Maria Brigida Ferraro, Paolo Giordani.

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http://dx.doi.org/10.1016/j.ijar.2017.06.009
0888-613X/

Please cite this article in press as: P. Berti et al., Basic ideas underlying conglomerability and disintegrability, Int. J. Approx. Reason. (2017), http://dx.doi.org/10.1016/j.ijar.2017.06.009

case. Indeed, the general theory of conglomerability/disintegrability is essentially understood in the precise case, while it is still in progress in the imprecise case.

To deal with conglomerability/disintegrability implies some choices about the ingredients of the problem. In this paper, for the reasons explained in Subsection 2.1, we restrict to bounded random variables. At least in the precise case, probability measures are finitely additive and conditioning is based on de Finetti's coherence principle. Finally, integrals are meant in Dunford–Schwartz's sense [2]; see Subsection 2.1 again.

2. Basics

2.1. Integral and notational conventions

Throughout, Ω is a non-empty set and Π a partition of Ω. For any set I, we let $\mathcal{P}(I)$ denote the power set of I and $l^\infty(I)$ the class of real bounded functions on I. Moreover, the abbreviation *f.a.p.* stands for *finitely additive probability*.

A random variable is a real function on Ω (no measurability constraints are required). In this paper, for convenience, we restrict to *bounded* random variables. The main reason for that is the integral representation. In fact, a coherent function on a class $\mathcal{D}$ of bounded random variables can be written as the integral with respect to a finitely additive probability, but this useful fact is no longer true if $\mathcal{D}$ includes unbounded random variables; see [3,4] and references therein. It should be stressed, however, that conglomerability/disintegrability can be developed for unbounded random variables as well. It suffices to broaden the notion of integral; see [4] again.

Since we restrict to bounded integrands, and since any bounded function is the uniform limit of simple functions, integrals with respect to f.a.p.'s are meant in the obvious way; see e.g. [2]. Fix in fact $X \in l^\infty(\Omega)$ and a f.a.p. μ on the σ-field generated by X. If X is simple, say $X = \sum_i a_i 1_{A_i}$, then $\int_\Omega X\,d\mu = \sum_i a_i\,\mu(A_i)$. Otherwise, $\int_\Omega X\,d\mu = \lim_n \int_\Omega X_n\,d\mu$ where X_n is a sequence of simple functions such that $X_n \to X$ uniformly.

We always denote by $\mathcal{D} \subset l^\infty(\Omega)$ a class of bounded random variables. In addition, three notational conventions are adopted. First, a set and its indicator are denoted by the same symbol. Thus, if $A \subset \Omega$, then A also designates the indicator function of the set A. Second, if $\mathcal{A}$ is a collection of subsets of Ω, we write $\mathcal{D} = \mathcal{A}$ (or $\mathcal{D} \supset \mathcal{A}$) to mean that $\mathcal{D}$ coincides with (or $\mathcal{D}$ includes) the class of indicators of the members of $\mathcal{A}$. Third, for each $S \subset \Pi$, we let S^* denote the subset of Ω obtained as union of the elements of S, namely, $S^* = \bigcup_{H \in S} H$. Roughly speaking, S and S^* are essentially the same set, but S is a subset of Π while S^* a subset of Ω.

2.2. Coherent conditional probabilities

Some claims in this subsection are without proofs. We refer to [5] and [6] for the latter.

A conditional bounded random variable, $X|H$, is the restriction of $X \in l^\infty(\Omega)$ to a (non-empty) subset $H \subset \Omega$. As usual, if $H = \Omega$, we write X instead of $X|\Omega$. Let $\mathcal{C}$ be any class of conditional bounded random variables and let P be a real function on $\mathcal{C}$. Then, P is *coherent* if, for all $n \geq 1$, $c_1, \ldots, c_n \in \mathbb{R}$ and $X_1|H_1, \ldots, X_n|H_n \in \mathcal{C}$, one obtains

$$\sup G|H \geq 0$$

where

$$G = \sum_{i=1}^n c_i\, H_i \left\{X_i - P(X_i|H_i)\right\} \quad \text{and} \quad H = \bigcup_{i=1}^n H_i. \tag{1}$$

This is de Finetti's coherence principle. Indeed, for an arbitrary class $\mathcal{C}$ of conditional bounded random variables, de Finetti's ideas have been realized by various authors independently; see [5–7] and references therein.

Some more remarks are in order.

A coherent function P is also called a *prevision*. It is called a *conditional probability* when each element of $\mathcal{C}$ is of the form $A|H$, with $A, H \subset \Omega$ and $H \neq \emptyset$.

Since $c_1, \ldots, c_n$ are arbitrary constants, one also obtains

$$\inf G|H = -\sup -G|H \leq 0$$

whenever P is coherent and G and H are given by (1).

If P is coherent, the map $X \mapsto P(X|H)$ is a linear positive functional supported by H. More precisely, fix a non-empty subset $H \subset \Omega$ and define $\mathcal{C}_H = \{X \in l^\infty(\Omega) : X|H \in \mathcal{C}\}$. Then,

$$\sup X|H \geq P(X|H) \geq \inf X|H \quad \text{and} \quad P\big(aX + bY|H\big) = a\,P(X|H) + b\,P(Y|H)$$

whenever $a, b \in \mathbb{R}$ and $X, Y, aX + bY$ belong to $\mathcal{C}_H$. In particular,

$$P(H|H) = 1 \qquad \text{provided } H \in \mathcal{C}_H.$$

Coherence of P admits various characterizations under some assumptions on $\mathcal{C}$. We just mention three cases. Let $\mathcal{A}$ be a field of subsets of Ω. Then,

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