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Logics with lower and upper probability operators [☆]

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ABSTRACT

We present a first-order and a propositional logic with unary operators that speak about upper and lower probabilities. We describe the corresponding class of models, and we discuss decidability issues for the propositional logic. We provide infinitary axiomatizations for both logics and we prove that the axiomatizations are sound and strongly complete. For some restrictions of the logics we provide finitary axiomatic systems.[1]

1. Introduction

During the last few decades, uncertain reasoning has emerged as one of the main fields in computer science and artificial intelligence. Many different tools are developed for representing and reasoning with uncertain knowledge. One particular line of research concerns the formalization in terms of probabilistic logic. After Nilsson [33] gave a procedure for probabilistic entailment that, given probabilities of premises, calculates bounds on the probabilities of the derived sentences, researchers from the field started investigations about formal systems for probabilistic reasoning [11,10,12,14,17,20,31,34,35].

However, in many applications, sharp numerical probabilities appear too simple for modeling uncertainty. This calls for developing different imprecise probability models [4,9,29,32,41,43–45]. In order to model some situations of interest, some approaches use sets of probability measures instead of one fixed measure, and the uncertainty is represented by two boundaries – lower and upper probabilities [22,28]. Consider the following example, essentially taken from [18].

Example 1. Suppose that a bag contains 10 marbles and we know that 4 of them are red, and the remaining 6 are either black or green, but we do not know the exact proportion (for example, it is possible that there are no green marbles at all). The goal is to model a situation where the person picks a marble from the bag at random. The cases when person picks up a red marble (red event), when person picks up a black marble (black event) and when person picks up a green marble (green

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event) will be denoted by R, B and G, respectively. Clearly, the probability of the red event is 0.4, but we cannot assign strict probability to black or green event. Therefore, we use the set of probability measures $P = \{\mu_\alpha \mid \alpha \in [0, 0.6]\}$, where μ_α assigns 0.4 probability to red event, α to black event, and $0.6 - \alpha$ to green event. We assign two functions to arbitrary set of probability measures P, first one is $P^\star(X) = sup\{\mu(X) \mid \mu \in P\}$ and the second one is $P_\star(X) = inf\{\mu(X) \mid \mu \in P\}$ which will be used to define a range of probabilities, i.e. they will be an upper and a lower probability, respectively.

One of the main problems in probabilistic logics with non-restricted real-valued semantics is that those formalisms are rich enough to express the type of a proper infinitesimal $\{0 < x < \frac{1}{n} \mid n = 1, 2, 3, \ldots\}$, so the logics are not compact (see Example 10). As an unpleasant logical consequence, for any finitary axiomatic system, there are consistent sets of formulas which are unsatisfiable [42], i.e., the axiomatization is not strongly complete. Halpern and Pucella [18] provided a finitary axiomatization for propositional reasoning about linear combinations of upper probabilities, and they proved weak completeness (every consistent formula is satisfiable) for the logic. Their formulas are Boolean combinations of the expressions of the form $r_1\ell(\alpha_1) + \cdots + r_n\ell(\alpha_n) \geq r_{n+1}$, where ℓ is the upper probability operator and r_i are real numbers,[2] for $i \in \{1, 2, \ldots, n+1\}$.

In this paper, we propose sound and strongly complete (every consistent set of formulas is satisfiable) propositional logic for reasoning about lower and upper probabilities (LUPP), and its first-order extension ($LUPFO$[3]). Our syntax is simpler than the one by Halpern and Pucella [18], since we don't have the arithmetical operations built into syntax. We extend propositional calculus (in the case of LUPP) and first-order languages (in the case of LUPFO) with modal-like unary operators of the form $U_{\geq s}$ and $L_{\geq s}$, where s ranges over the unit interval of rational numbers. The intended meanings of $U_{\geq s}\alpha$ and $L_{\geq s}\alpha$ are "the upper/lower probability of α is at least s". If $Green$ is a propositional letter, then by the $LUPP$ formula $U_{=0.6}Green$ is a $LUPP$ we can represent the fact that upper probability that a green marble is picked is 0.6 (Example 1). Note that we can also represent the strict probability of choosing a red marble (Red) in LUPP. The formula which assigns the probability 0.4 to that event is $L_{\geq 0.4}Red \wedge U_{\leq 0.4}Red$. In the first-order case, we consider formulas like $L_{\geq 0.5}(\forall x)R(x)$. In natural language, this sentence can represent the statement "the lower probability that it will rain in all the regions of the considered country is at least a half." (Here the lower probability can arise from considering different weather reports, where each report assigns fixed chance of rain to every region and also a fixed chance for raining in all regions.) Now we introduce another example that can be modeled in $LUPFO$.

Example 2. Authorities of a certain country are worried that Zika virus may be carried into the country across the borders, and have engaged a number of health experts to estimate the probability of at least one infected person entering the country. If the highest estimated probability is over a certain threshold tr, the authorities will institute a restricted border crossing regime. That constraint can be represented in our logic by the formula

$$U_{\geqslant tr}(\exists x)\text{Zika}(x).$$

The corresponding semantics of our logics consist of special types of Kripke models (possible worlds). In the propositional case, each possible world contains an evaluation of propositional letters, while in the first-order case it contains a first-order structure of a chosen language. In addition, each model is equipped with a set of probability measures defined over the worlds. In order to obtain strong completeness, we use infinitary inference rules. Thus our languages are countable and formulas are finite, while only proofs are allowed to be infinite. We also propose the restricted logics $LUPP^{FR(n)}$ and $LUPFO^{FR(n)}$[4] (for each n in $\mathbb{N} \setminus \{0\}$). For those logics, we achieve compactness using only a finite set of probability values, which is still enough for many practical applications. We propose finitary axiomatization for $LUPP^{FR(n)}$ and $LUPFO^{FR(n)}$.

From the technical point of view, we have modified some of our earlier developed completion methods presented in [5–8,23,24,34–36,39]. Providing a compete axiom system for the logic is the key issue in formalization of reasoning about upper and lower probabilities. In real-world situations, we usually don't have the complete specifications of systems, but we can obtain probability constrains from different sources. In that way we derive upper and lower probabilities, and the complete axiom system provides tools to deduce formal properties of the considered system.

The contents of this paper are as follows. In Section 2 we recall the notions of lower and upper probability, as well as the representation theorem we use in our axiomatizations. In Section 3 we present the syntax and semantics of the logic $LUPFO$, as well as the axiomatization and we prove some auxiliary propositions. We prove the soundness and completeness of the axiomatization in Section 4. In Section 5 we introduce the logic $LUPP$ and discuss its decidability. We comment the axiomatization of the logic $LUPP$ in Section 6, and the soundness and completeness theorem for that logic in Section 7. In Section 8 we present the finitary logics $LUPP^{FR(n)}$ and $LUPFO^{FR(n)}$, where the probabilities are restricted to a finite set. Section 9 is dedicated to related work and we conclude in Section 10.

2 Halpern and Pucella [18] define the rich language with formulas with all the reals as coefficients. But, in order to obtain decidability, they have to restrict their language and allow only integer coefficients, i.e. $r_i \in \mathbb{Z}$.

3 LUP stands for "lower and upper probabilities". The suffixes P and FO indicate that the logic is propositional or first-order, respectively.

4 FR stands for "finite range".