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From propositional logic to plausible reasoning: A uniqueness theorem

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ABSTRACT

We consider the question of extending propositional logic to a logic of plausible reasoning, and posit four requirements that any such extension should satisfy. Each is a requirement that some property of classical propositional logic be preserved in the extended logic; as such, the requirements are simpler and less problematic than those used in Cox's Theorem and its variants. As with Cox's Theorem, our requirements imply that the extended logic must be isomorphic to (finite-set) probability theory. We also obtain specific numerical values for the probabilities, recovering the classical definition of probability as a *theorem*, with truth assignments that satisfy the premise playing the role of the "possible cases."

1. Introduction

E.T. Jaynes [13, p. xxii] proposes the view that probability theory is the *uniquely determined* extension of classical propositional logic (CPL) to a "logic of plausible reasoning":

> Our theme is simply: *probability theory as extended logic*. ...the mathematical rules of probability theory are not merely rules for calculating frequencies of 'random variables'; they are also the unique consistent rules for conducting inference (i.e. plausible reasoning) of any kind...

This view is grounded in the work of Pólya [17] and Cox [9], especially the latter. In this paper we aim to set the notion of probability theory as the necessary extension of CPL on solid footing.

Our goal is to generalize the logical consequence relation, which deals only in certitudes, to handle *degrees* of certainty. Whereas $X \models A$ means that A (the conclusion) is a logical consequence of X (the premise), we write $A \mid X$ for "the reasonable credibility of the proposition A (the query) when the proposition X (the premise) is known to be true" (paraphrasing Cox [9]). We call $(\cdot \mid \cdot)$ the *plausibility function*. If $X \models A$ then $A \mid X$ is some value indicating "certainly true," if $X \models \neg A$ then $A \mid X$ is some value indicating "certainly false," and otherwise $A \mid X$ is a value indicating some intermediate level of plausibility. Our task is to determine what the plausibility function must be, based on logical criteria.

As a generalization of the logical consequence relation, the plausibility function must depend *only* on its two explicit arguments; the value it returns must not depend on any additional information that varies according to the problem domain to which it is applied, nor according to the intended meanings of the propositional symbols. This is a *formal logical* theory

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we are developing, and so any intended semantics of the propositional symbols must be expressed axiomatically in the premise.

One might question whether all relevant information for determining the plausibility of some proposition A can be expressed in propositional form for inclusion in the premise X. Might not our background information include "soft" relationships, mere *propensities* for propositions to be associated in some way? Although we provide some suggestive examples, we do not attempt to resolve that question. Instead we ask, *given* that the background information and intended semantics are expressed in propositional form and included in the premise, with no other information available, what can we conclude about the plausibility function?

We posit four Requirements for the plausibility function. Each of these requires that some property of the logical consequence relation be retained in the generalization to a plausibility function. Three are invariance properties, and the fourth is a requirement to preserve distinctions in degree of plausibility that already exist within CPL. These Requirements (discussed in detail later) are the following:

R1. If X and Y are logically equivalent, and A and B are logically equivalent assuming X, then $A \mid X = B \mid Y$ (Section 4).
R2. We may define a new propositional symbol without affecting the plausibility of any proposition that does not mention that symbol. Specifically, if s is a propositional symbol not appearing in A, X, or E, then $A \mid X = A \mid (s \leftrightarrow E) \wedge X$ (Section 5).
R3. Adding irrelevant information to the premise does not affect the plausibility of the query. Specifically, if Y is a satisfiable propositional formula that uses no propositional symbol occurring in A or X, then $A \mid X = A \mid Y \wedge X$ (Section 6).
R4. The implication ordering is preserved: if $X \models A \rightarrow B$ but not $X \models B \rightarrow A$ then $A \mid X$ is a plausibility value that is strictly less than $B \mid X$ (Section 7).

Note that we do *not* assume that plausibility values are real numbers, nor that they are totally ordered; R4 presumes only that there is some *partial* order on plausibility values.

Given R1–R4, we prove that plausibilities are essentially probabilities in disguise. Specifically, we show that

1. there is an order-preserving isomorphism P between the set of plausibility values $\mathbb{P}$ and the set of rational probabilities $\mathbb{Q} \cap [0, 1]$;
2. $P(A \mid X)$, the plausibility $A \mid X$ mapped via P to the unit interval, is *necessarily* the ratio of the number of truth assignments that satisfy both A and X to the number of truth assignments that satisfy X; and
3. hence the usual laws of probability follow as a consequence.

This identifies finite-set probability theory as the uniquely determined extension of CPL to a logic of plausible reasoning.

The body of this paper is organized as follows:

- In Section 2 we compare this work to Cox's Theorem and variants, as well as Carnap's system of logical probability.
- In Section 3 we review some notions from CPL, discuss the partial plausibility ordering that *already exists* within CPL, and discuss the nature of the plausibility function.
- Our main result is proven in Sections 4, 5, 6, and 7, which also introduce the Requirements, discuss the motivation behind them, and explore some of their consequences. Along the way we discuss how Carnap's system violates R3.
- In Section 8 we prove that R1–R4 are consistent.
- Section 9 discusses three topics: the connection of our results to the classical definition of probability, the issue of non-uniform probabilities, and an initial attempt at extending our results to infinite domains.

2. Relation to prior work

To set the context for this paper, clarify our goals, and head off possible misconceptions, we now review similar prior work and point out the differences.

2.1. Cox's theorem

R.T. Cox [9] proposes a handful of intuitively-appealing, qualitative requirements for any system of plausible reasoning, and shows that these requirements imply that any such system is just probability theory in disguise. Specifically, he shows that there is an order isomorphism between plausibilities and the unit interval $[0, 1]$ such that $A \mid X$, after mapping from plausibilities to $[0, 1]$, respects the laws of probability.

Over the years Cox's arguments have been refined by others [1,13,16,20,21], making explicit some requirements that were only implicit in Cox's original presentation, and replacing some of the requirements with slightly less demanding assumptions than those used in Cox's original proof. One version of the requirements [21] may be summarized as follows:

C1. $A \mid X$ is a real number.
C2. $A \mid X = A' \mid X$ whenever A is logically equivalent to A', and $B \mid X \leq A \mid X$ for any tautology A.