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A Banzhaf value for games with a proximity relation among the agents

J.R. Fernández, I. Gallego, A. Jiménez-Losada *, M. Ordóñez

Escuela Técnica Superior de Ingenieros, Matemática Aplicada II, Universidad de Sevilla, Camino de los Descubrimientos, 41092 Sevilla, Spain

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ABSTRACT

The Banzhaf index is a function determining the power or influence in the decision of a set of agents. The extension of this index to the family of the cooperative games is named Banzhaf value. The relationships of closeness among the agents should modify their power. Games with a priori unions study situations where the closeness relations among the agents are taken into account. In this model the agents are organized in an a priori partition where each element of the partition represents a group of agents with close interests or ideas. The power is determined in two steps, first as a problem among the unions and later, inside each one, the power of each agent is determined. Proximity relations extend this model considering leveled closeness among the agents. In this paper we analyze a version of the Banzhaf value for games with a proximity relation and we show the interest of this value by applying it to the allocation of the power of the political groups in the European Parliament.

1. Introduction

In decision situations for committees or centers of distributed control the quantification of the power of each member is an important element to analyze the final position and the different treatment of each of them. Simple games are a way from the cooperative game theory to represent these situations and to study the power of their elements. A power index for simple games is a function determining the power or influence of the agents in each simple game. One of the most known power indices, the Banzhaf index, was introduced by Penrose [1] in 1946 and later by Banzhaf [2] in 1965. In this context the Banzhaf index was generalized for all cooperative game as the Banzhaf value [3]. The Banzhaf value has been studied in different scenarios incorporating new information over the relationships of the agents (coalition structures [4], communication situations [5], hierarchical relations [6], etc.). Owen [7] proposed a different model with an evident interest for simple games. He considered that agents are organized a priori in groups taking into account the closeness of their interests (ideas). So, besides the game he supposed known a partition of the set of agents in a priori unions based in the relations among the agents. These unions are considered as a starting point for further negotiations. This model allows to determine the power of the agents in a simple game taking into account the a priori unions among them. But closeness is usually a leveled property. For instance, political groups can be organized in a priori ideological unions. The Banzhaf value was studied for this model in [8]. Casajus [9] proposed another version of the Owen model considering also information about the internal estructure of the unions. But the Banzhaf value has not been studied for this version.

* Corresponding author.
E-mail address: ajlosada@us.es (A. Jiménez-Losada).

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Considering equal every ideological closeness between two political parties is actually a simplification of the situation. Aubin [10] and Butnariu [11] introduced fuzzy sets to describe leveled participation of the players in the coalitions (fuzzy coalitions). Proximity relations are reflexive and transitive fuzzy binary relations. Fernández et al. [12] introduced proximity relations to explain the relations among the players in a cooperative game, extending the Owen model in a natural way for another known value, the Shapley value. Related works are given in Meng [13] and Meng and Zhang[14], but they consider a game with fuzzy coalitions with a crisp system of a priori unions. Hence this model contains a different approach. Others analyze this kind of relations in a probabilistic way, Calvo et al. [15] or Kaniovski and Das [16], but they do not use the Owen model. Now, we propose to use proximity relations to study the power of an agent in a game by the Banzhaf value. Section 2 is dedicated to the preliminaries about cooperative games, a priori unions and fuzzy sets. In section 3 we introduce our Banzhaf value for games with a proximity relation among the agents and particularly the Banzhaf–Owen value is extended to the Casajus version of the Owen model. In section 4 we propose axioms for the proposed value extending known properties of the classical Banzhaf value to our fuzzy situation. Finally, last section shows an example of application of the model as index, analyzing the power of the political groups in the European Parliament in an ad hoc situation.

2. Preliminaries

2.1. Cooperative games

A *cooperative game with transferable utility*, game from now on, is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a mapping satisfying $v(\emptyset) = 0$. The elements of N are named players, the subsets of players are named coalitions and the mapping v is the characteristic function of the game. A *simple game* represents a decision situation by a cooperative game (N, v) where: 1) $v(S) \in \{0, 1\}$ for every $S \subseteq N$, 2) $v(N) = 1$, and 3) $v(S) \leq v(T)$ if $S \subset T \subseteq N$ (monotonicity). A coalition is called winning if $v(S) = 1$ and losing $v(S) = 0$.

A *value* for games is a function ψ which determines for each game (N, v) a vector $\psi(N, v) \in \mathbb{R}^N$ interpreted as a payoff vector. Values for simple games are named power indices.[1] In this case the payoffs mean the power or influence of the agents in the decision. This paper focuses on the Banzhaf value. A swing for a player $i \in N$ in a simple games is a winning coalition S containing player i such that $S \setminus \{i\}$ is losing. The Banzhaf index obtains the probability to get a swing among the coalitions containing a determined agent. Owen [3] extended this index to all the cooperative games. The *Banzhaf value* is a value defined for every $(N, v) \in \mathcal{G}$ and $i \in N$ as

$$\beta_i(N, v) = \sum_{\{S \subset N : i \notin S\}} \frac{1}{2^{|N|-1}} [v(S \cup \{i\}) - v(S)]. \tag{1}$$

2.2. Communication structures

Myerson [5] analyzed the inclusion in a game of information about the communication of the players. Let N be a finite set of players and $L^N = \{\{i, j\} \in N \times N : i \neq j\}$ the set of unordered pairs of different elements in N. We use $ij = \{i, j\}$ from now on. Each undirected graph (N, L) where the set of vertices is N and the set of edges $L \subseteq L^N$ is considered as a communication structure. So, each $L \subseteq L^N$ is called a *communication structure for* N. Myerson defines a *game with communication structure* as a triple (N, v, L) where (N, v) is a game and L is a communication structure for N. A usual cooperative game (N, v) is identified with the game with communication structure (N, v, L^N). Let (N, v, L) be a game with communication structure. A coalition $S \subseteq N$ whose vertices are connected by the links in L is called *connected*. The maximal connected coalitions correspond to the sets of vertices of the connected components of the graph (N, L) and we denote them as N/L. This family N/L is actually a partition of N. If $S \subseteq N$ is a coalition then $L_S = \{ij \in L : i, j \in S\}$ and (S, v, L_S) represents the restriction to S of the characteristic function of the game and the communication structure. We use $S/L = S/L_S$. Given (N, v, L), Myerson introduces a new game $(N, v/L)$ incorporating the information of the communication structure,

$$v/L(S) = \sum_{T \in S/L} v(T) \quad \forall S \subseteq N.$$

This model supposes then that non-connected coalitions do not obtain extra profits with regard to their components and so they are irrelevant. The Banzhaf value was extended for games with communication structure in [18]. The *graph-Banzhaf value* is a function defined by

$$\eta(N, v, L) = \beta(N, v/L). \tag{2$$

[1] This is a cardinal notion of power rather than other ordinal ones, see [17].