ARTICLE IN PRESS
IJA:7985

International Journal of Approximate Reasoning ••• (••••) •••–•••

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Fuzzy memberships as likelihood functions in a possibilistic framework

Giulianella Coletti [a], Davide Petturiti [a,*], Barbara Vantaggi [b]

[a] Dip. Matematica e Informatica, Università di Perugia, 06100 Perugia, Italy
[b] Dip. S.B.A.I., "La Sapienza" Università di Roma, 00185 Roma, Italy

ARTICLE INFO

Article history:
Received 31 March 2016
Received in revised form 13 November 2016
Accepted 25 November 2016
Available online xxxx

Keywords:
Conditional possibility
Conditional probability
Likelihood function
Inference
Fuzzy set

ABSTRACT

Likelihood functions are studied in a probabilistic and possibilistic setting: inferential conclusions are drawn from a set of likelihood functions and prior information relying on the notion of disintegrability. The present study allows for a new interpretation of fuzzy membership functions as coherent conditional possibilities. The concept of possibility of a fuzzy event is then introduced and a comparison with the probability of a fuzzy event is provided.

1. Introduction

Models and tools for jointly handling uncertainty and vagueness have to be performed in order to deal with problems involving several heterogeneous sources of knowledge. This fact generates new issues in probability and statistics, and so methods able to combine uncertainty measures and fuzzy information need to be studied.

To this aim likelihood functions have to be considered inside different frameworks. In this paper we refer to probabilistic and possibilistic settings and we consider likelihood functions with prior uncertainty measures. In the probabilistic framework this problem reduces to Bayes rule, in the simplest case. However, the hypotheses of Bayes theorem are not satisfied when, for example, the likelihood and prior information are defined on different spaces. Then, one needs to handle generalized Bayesian inferential procedures whose result is in general not unique, but consists in an interval of coherent values for any new conditional event. In these situations, the main theoretical problem is to provide a consistent framework where to merge all the available information. The notion of coherence has a central role since it guarantees an effective tool for controlling global consistency and ruling the inferential procedures, that essentially are extension problems maintaining consistency with the framework of reference.

The aim of this paper is to merge in the context of possibility theory both prior uncertainty quantified by a possibility measure (for instance obtained as upper envelope of the extensions of a probability related to a different variable [13,41]) and vagueness expressed by fuzzy sets. Recall that an interpretation of the fuzzy membership in terms of probabilistic likelihood has been given in [11,12,22,47], moreover, an analogous interpretation in terms of coherent lower and upper conditional probabilities can be found in [17].

* Corresponding author.
E-mail addresses: giulianella.coletti@unipg.it (G. Coletti), davide.petturiti@unipg.it (D. Petturiti), barbara.vantaggi@sbai.uniroma1.it (B. Vantaggi).

http://dx.doi.org/10.1016/j.ijar.2016.11.017
0888-613X/

Please cite this article in press as: G. Coletti et al., Fuzzy memberships as likelihood functions in a possibilistic framework, Int. J. Approx. Reason. (2017), http://dx.doi.org/10.1016/j.ijar.2016.11.017

Now, following the same line, an interpretation of a membership function as a coherent T-conditional possibility (where T is a continuous triangular norm) is proposed. The semantic behind this new interpretation follows the one of [11,12,17]: for every x in the range of a variable X, the value of the membership $\mu_\varphi(x)$ of a fuzzy set related to a property φ of X is *the measure of how much You believe in the Boolean event "You claim that X has property φ" when the variable X takes the value x*. One of the main reasons for interpreting the membership as a possibilistic likelihood resides in the semantic meaning of the membership itself (see [22]).

We investigate the main properties of a likelihood function, an aggregated likelihood function and their extensions. This requires to study inferential processes inside the possibilistic setting by considering any continuous t-norm. For that, we extend inferential results given for the minimum and strict t-norms in [6,7,15,16,27], and introduce coherent T-conditional possibility envelopes, with T a continuous t-norm.

In this framework, we check which operations between fuzzy sets arise when the memberships are regarded as possibilistic likelihoods. For that, we need to study the coherence of a class of possibilistic likelihood functions. Furthermore, a comparison between possibilistic and probabilistic interpretations of fuzzy memberships and their impact on the resulting fuzzy operations is drawn.

As discussed before, our main aim is to make inference starting from a "prior" possibility measure and a family of fuzzy membership functions viewed as possibilistic likelihood functions. To handle this procedure it is necessary, first of all, to check whether possibilistic and fuzzy information are globally coherent (i.e., consistent with respect to the chosen definition of conditioning) and then to extend the coherent assessment to new conditional events maintaining coherence. The aforementioned inferential procedure allows to compute the possibility of a "fuzzy event", so we get (as a particular case) Zadeh's definition [52] and, moreover, the conditional possibility of a "conditional fuzzy event".

Finally, we show how to create a collection of imprecise possibilistic fuzzy IF–THEN rules: in particular we discuss how to compute the possibility associated to these rules and how to propagate the possibilistic information, when either the premise or the consequence is fuzzy.

2. *T*-conditional possibilities

An *event* E is singled out by a Boolean proposition, that is a statement that can be either true or false. Since in general it is not known whether E is true or not, we are uncertain on the realization of E, which, in this case, is said to be *possible*. We denote with Ω and $\emptyset$ the *certain event* and the *impossible event*, respectively. Such events coincide with the top and the bottom of every Boolean algebra $\mathcal{A}$ of events, i.e., a set of events closed with respect to the Boolean operations of *contrary* ^{c}, *conjunction* $\wedge$ and *disjunction* $\vee$ and equipped with the partial order $\subseteq$ of *implication*. A *conditional event* $E|H$ is an ordered pair (E, H) of events, with $H \neq \emptyset$. In particular any event E can be seen as the conditional event $E|\Omega$.

In what follows, $\mathcal{A} \times \mathcal{H}$ denotes a set of conditional events with $\mathcal{A}$ a Boolean algebra and $\mathcal{H}$ an additive set (i.e., closed with respect to finite disjunctions) such that $\mathcal{H} \subseteq \mathcal{A}^0 = \mathcal{A} \setminus \{\emptyset\}$. Recall that any arbitrary set of conditional events $\mathcal{G} = \{E_i|H_i\}_{i\in I}$ can be embedded in a set $\mathcal{A} \times \mathcal{H}$ by taking $\mathcal{A} = \langle\{E_i, H_i\}_{i\in I}\rangle$ and $\mathcal{H} = \langle\{H_i\}_{i\in I}\rangle^{\mathbf{A}}$, which are the Boolean algebra generated by $\{E_i, H_i\}_{i\in I}$ and the additive class generated by $\{H_i\}_{i\in I}$, respectively.

In literature several definitions of conditional possibility have been introduced [1,2,4,18,21,33,51], highlighting that the operation of conditioning is quite controversial. In the majority of the definitions above, a conditional possibility $\Pi(E|H)$ is defined as a solution of the equation

$$\Pi(E \wedge H) = T(x, \Pi(H)), \tag{1}$$

where T is a t-norm [36]. This equation necessarily admits a solution when T is continuous, but the continuity of T is not sufficient to guarantee the uniqueness of the solution. In fact, to get a unique solution some further requirement has been assumed in literature, such as the *minimum specificity principle* [23], which consists in selecting the greatest solution of equation (1).

In the following we refer to the notion of T-conditional possibility (with T any t-norm) introduced in [4,16].

Definition 1. Let T be any t-norm. A function $\Pi : \mathcal{A} \times \mathcal{H} \to [0, 1]$ is a T**-conditional possibility** if it satisfies the following properties:

(CP1) $\Pi(E|H) = \Pi(E \wedge H|H)$, for every $E \in \mathcal{A}$ and $H \in \mathcal{H}$;
(CP2) $\Pi(\cdot|H)$ is a finitely maxitive possibility on $\mathcal{A}$, for any $H \in \mathcal{H}$;
(CP3) $\Pi(E \wedge F|H) = T(\Pi(E|H), \Pi(F|E \wedge H))$, for any $H, E \wedge H \in \mathcal{H}$ and $E, F \in \mathcal{A}$.

Following the terminology of [20] for conditional probabilities, a T-conditional possibility is said *full on* $\mathcal{A}$ if $\mathcal{H} = \mathcal{A}^0$, i.e., if its domain is $\mathcal{A} \times \mathcal{A}^0$.

Let us stress that condition *(CP2)* requires that, for every $H \in \mathcal{H}$, $\Pi(\cdot|H)$ is a normalized finitely maxitive function [46] defined on $\mathcal{A}$, i.e., it satisfies

Please cite this article in press as: G. Coletti et al., Fuzzy memberships as likelihood functions in a possibilistic framework, Int. J. Approx. Reason. (2017), http://dx.doi.org/10.1016/j.ijar.2016.11.017