International Journal of Approximate Reasoning 81 (2017) 147–159

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

An inquiry into approximate operations on fuzzy numbers

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ARTICLE INFO

Article history:
Received 31 August 2016
Accepted 14 November 2016
Available online 28 November 2016

Keywords:
Fuzzy numbers
Fuzzy arithmetics
Extension principle
Approximate operations

ABSTRACT

Operations on fuzzy numbers have been a cornerstone in the development of fuzzy modeling and computing with words. Although exact operations are commonly defined by the extension principle, many applications employ approximate operations. At present, despite their wide use, there is no evidence on the goodness of approximate operations. By means of both numerical simulations and theoretical results, in this paper we present an analysis of approximate operations on fuzzy numbers. By focusing on the ranking and defuzzification procedures as essential tools in fuzzy decision making problems, we are going to study the errors produced by the application of approximate operations.

1. Introduction

For the last fifty years, fuzzy sets theory, also thanks to its connection with possibility theory, has successfully been used to model various phenomena involving imprecision and uncertainty. The efficacy and the soundness of fuzzy sets and fuzzy numbers have been advocated in operations research [26], decision sciences [13], and many other branches of social and hard sciences. It is in these contexts that fuzzy numbers have been profitably used to represent uncertain parameters, variables and arguments of functions. For this reason, and because fuzzy numbers represent generalizations of real numbers, an appropriate extension of the concept of function has been necessary to make mathematical models work when fuzzy numbers, instead of real numbers, are used. The definition of such extensions has triggered the development of fuzzy real algebra and arithmetics [15].

The most widely accepted methodology to extend the concept of function to work with fuzzy sets is based on the so-called extension principle by Zadeh [28], sometimes also called fuzzification principle [14]. If we focus our attention on fuzzy numbers with piecewise linear membership function, then, by the extension principle, we obtain that the operations of addition and subtraction preserve linearity. On the contrary, when the extension principle is used to calculate the product and the quotient of two piecewise linear fuzzy numbers, the resulting membership function is *not* piecewise linear. In spite of the nonlinearity of the results, for sake of simplicity or lack of knowledge of the correct principles, in many applications approximate operations preserving the linear shape are used. Examples of applications where approximations were used are numerous and encompass data analysis [10], risk analysis [11], investment evaluation [12], product positioning [23], decision making [33], logistics [18], and corporate finance [25], just to cite a few of them.

In spite of the wide use of approximate operations, it seems that researchers have not focused their attention on the analysis of the errors that can be made by using them. Naturally, it follows that approximate operations are presently being used without any real awareness of their capacity (or incapacity) of approximating correct operations. In the literature

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http://dx.doi.org/10.1016/j.ijar.2016.11.011

there are very few studies on this issue. In their paper on operations on fuzzy numbers Dubois and Prade [14] warned the community from the dangers associated with approximate operations. As for a numerical study on the errors, the literature offers nothing but a study by Giacchetti and Young [17] which, however, showed only some specific examples for the case of the multiplication. Thus, it seems important to study whether or not approximate operations are suitable and reliable alternatives for the operations defined by the extension principle.

This said, the scope of this study is precisely that of filling this lack of research on the errors generated by the use of approximate operations. In this spirit, by means of both numerical simulations and theoretical results, we shall analyze the magnitude of the errors made while resorting to approximate operations.

The paper is organized as follows. Section 2 contains the necessary notations, the definitions of operations on fuzzy numbers, and recalls the ranking methods for fuzzy numbers which are going to be used later in the analysis. Section 3 presents the problem and the methodology used in the numerical simulations, whereas the results are presented and interpreted in Section 4. To corroborate the results prompted by the numerical simulations, Section 5 provides some formal results on the extent of the possible error generated by approximate operations. Finally, Section 6 discusses the results and their implications in a broader sense.

2. Arithmetic operations and ranking of fuzzy numbers

Given a universal set X, a fuzzy set A is a set of pairs $\{(x, \mu_A(x))|x \in X\}$ where $\mu_A(x) \in [0, 1]$ is the degree of membership of x in A. For sake of simplicity, as it is customary to do, hereafter $\mu_A(x)$ is shortened as $A(x)$. Fixing an $\alpha \in [0, 1]$, the α-cut of a fuzzy set A is the set $A_\alpha = \{x \in X|A(x) \geq \alpha\}$. A fuzzy set is *convex* if and only if all its α-cuts are convex sets. A fuzzy set A is *normal* if and only if there exists an $x_0 \in X$ such that $A(x_0) = 1$. A *fuzzy number* is a normal and convex fuzzy set on the real line. The *support* of a fuzzy number is defined as $cl\{x \in A|A(x) > 0\}$ where cl denotes the topological closure of a set. The *core* of A is the α-cut with $\alpha = 1$; that is, the set of $x \in X$ with membership function value in A equal to 1. Since for any fuzzy number A all the α-cuts are intervals, one can define them by means of their endpoints $A^+_\alpha = \sup A_\alpha$ and $A^-_\alpha = \inf A_\alpha$.

Ban and Coroianu [4] correctly pointed out that "in most applications the researchers use only triangular or trapezoidal fuzzy numbers". Hence, for practical purposes it is convenient to consider these types of fuzzy numbers. For reasons which are going to be explained later, in this study we focus on triangular fuzzy numbers. A *triangular fuzzy number* A is defined by the following membership function,

$$A(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & \text{if } a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & \text{if } a_2 \leq x \leq a_3 \\ 0, & \text{otherwise} \end{cases}$$

with $a_1 \leq a_2 \leq a_3$. Note that there is a one-to-one correspondence between triangular fuzzy numbers and triples (a_1, a_2, a_3) with $a_1 \leq a_2 \leq a_3$ and therefore, for brevity, we shall write $A = (a_1, a_2, a_3)$. The second value, a_2, is usually called the *center* of the triangular fuzzy number. A special case of the definition, which will arise later in the numerical analysis, is when $a_1 = a_2$. We will call this case as *right-shoulder triangular fuzzy number*.

2.1. Arithmetic operations on fuzzy numbers

One fundamental concept in fuzzy sets theory concerns the idea of function $f : A_1 \times \cdots \times A_n \to B$ when $A_1, \ldots, A_n$ are fuzzy sets. That is, given a function f and a fuzzy domain $A_1 \times \cdots \times A_n$, one might inquire on the nature of B. The most widely agreed way of defining the membership function of B is by means of the *extension principle* [41], according to which

$$B(y) = \begin{cases} \sup_{y=f(x_1,\ldots,x_n)} \min\{A_1(x_1), \ldots, A_n(x_n)\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $f^{-1}(y)$ is the preimage of y. Consequently, considering two fuzzy numbers A_1 and A_2, the four arithmetic operations in two variables are defined as

$$(A_1 + A_2)(y) = \sup_{y=x_1+x_2} \min\{A_1(x_1), A_2(x_2)\} \tag{2}$$

$$(A_1 - A_2)(y) = \sup_{y=x_1-x_2} \min\{A_1(x_1), A_2(x_2)\} \tag{3}$$

$$(A_1 * A_2)(y) = \sup_{y=x_1*x_2} \min\{A_1(x_1), A_2(x_2)\} \tag{4}$$

$$(A_1 / A_2)(y) = \sup_{y=x_1/x_2} \min\{A_1(x_1), A_2(x_2)\} \tag{5}$$