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Self-dual operators and a general framework for weighted nilpotent operators

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ABSTRACT

The main purpose of this paper is to consider generated nilpotent operators in an integrative frame and to examine the nilpotent aggregative operator. As a starting point, instead of associativity, we focus on the necessary and sufficient condition of the self-dual property. A parametric form of the generated operator o_ν is given by using a shifting transformation of the generator function. The parameter has an important semantical meaning as a threshold of expectancy (decision level). Nilpotent conjunctive, disjunctive, aggregative and negation operators can be obtained by changing the parameter value. The properties (De Morgan property, commutativity, self-duality, fulfillment of the boundary conditions, bisymmetry) of the weighted general operator are examined and the formula of the commutative self-dual generated operator, the so-called weighted aggregative operator is given. It is proved that the two-variable operator with weights $w_1 = w_2 = 1$ $\forall i$ is conjunctive for low input values, disjunctive for high ones, and averaging otherwise; i.e. a high input can compensate for a lower one.

1. Introduction

One of the most significant problems of fuzzy set theory is the proper choice of set-theoretic operations [29,32]. Triangular norms and conorms have been thoroughly examined in the literature [14,15,19,22], and are often used as conjunctions and disjunctions in logical structures [18,27].

The most well-characterized class of t-norms is the so-called representable t-norms. t-norms generated by continuous additive generators were described by Mostert and Shield [26]. The two main types of representable t-norms are the strict and non-strict or nilpotent t-norms. The nilpotent operators have some nice properties which make them more useful when constructing logical structures. Among these properties are the fulfillment of the law of contradiction and the excluded middle, and the coincidence of the residual and the S-implication [11,31]. In [8], Dombi and Csiszár showed that a consistent connective system generated by nilpotent operators is not necessarily isomorphic to the Łukasiewicz-system. Using more than one generator function, consistent nilpotent connective systems (so-called bounded systems) can be obtained in a significantly different way with three naturally derived negation operators. Due to the fact that all continuous Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm [19], the previously studied nilpotent systems were all isomorphic to the well-known Łukasiewicz-logic. In [9] and in [10], Dombi and Csiszár examined the implications and equivalence operators in bounded systems.

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In human thinking, averaging operators, where a high input can compensate for a lower one, play a significant role. The aggregative operator was first introduced in 1982 by Dombi [7], by selecting a set of minimal concepts that must be fulfilled by an evaluation-like operator. The concept of uninorms was introduced in [33], as a generalization of both t-norms and t-conorms. By adjusting its neutral element ν, a uninorm is a t-norm if $\nu = 1$ and a t-conorm if $\nu = 0$. Uninorms have turned out to be useful in many areas like expert systems [6], aggregation [3,34] and the fuzzy integral [4,21].

The main difference in the definition of the uninorms and aggregative operators is that the self-duality requirement does not appear in uninorms, and the neutral element property is not in the definition for the aggregative operators. The representation theorem for strict, continuous on $[0,1] \times [0,1] \setminus \{(0,1),(1,0)\}$ uninorms (or representable uninorms) was given by Fodor et al. [16] (see also Klement et al. [20]). Such uninorms are called representable uninorms and they were previously introduced as aggregative operators [7]. Recently, a characterization of the class of uninorms with a strict underlying t-norm and t-conorm was presented in [13]. In [24], the authors show that uninorms with nilpotent underlying t-norm and t-conorm belong to U_{min} or U_{max}. Further results on uninorms with fixed values along their borders can be found in [5].

Our main purpose here is to consider generated nilpotent operators in an integral frame and to examine the nilpotent self-dual generated operators. A general parametric framework for the nilpotent conjunctive, disjunctive, aggregative and negation operators is given and it is demonstrated how the nilpotent generated operator can be applied for preference modeling.

The article is organized as follows. After a preliminary discussion in Section 2, a general parametric operator $o_\nu(\mathbf{x})$ of nilpotent systems is given in Section 3. The parameter has an important semantical meaning as the threshold of expectancy. In Section 4, the weighted form of this operator, $a_{\nu,\mathbf{w}}(\mathbf{x})$ is examined. In Section 5, the properties (De Morgan property, commutativity, self-duality, fulfillment of the boundary conditions, bisymmetry) of the weighted general operator are examined. Here, the formula for the commutative self-De Morgan operator, the so-called weighted aggregative operator is presented. Then in Section 6 we focus on the two-variable case, where it is proved that the two-variable operator with weights $w_1 = w_2 = 1$ is conjunctive for low input values, disjunctive for high ones, and averaging otherwise; i.e. a high input can compensate for a lower one. In Section 7, the main results are summarized and a possible direction of future work is mentioned.

2. Preliminaries

2.1. Negations, t-norms and t-conorms

First, we recall some basic notations and results regarding negation operators, t-norms and t-conorms that will be useful in the sequel.

Definition 1. A unary operation $n : [0,1] \to [0,1]$ is called a negation if it is non-increasing and compatible with classical logic; i.e. $n(0) = 1$ and $n(1) = 0$.

A negation is strict if it is also strictly decreasing and continuous.

A negation is strong, if it is also involutive; i.e. $n(n(x)) = x$.

The well-known representation theorem for strong negations was obtained by Trillas in [30]:

Proposition 1. *$n(x) : [0,1] \to [0,1]$ is a strong negation if and only if there exists an increasing bijection $f_n(x) : [0,1] \to [0,1]$ such that*

$$n(x) = f_n^{-1}(1 - f_n(x)).$$

Remark 1. In Proposition 1, the bijection may also be decreasing (see Dombi and Csiszár [8]).

Definition 2. Let $o(x, y) : [0,1]^2 \to [0,1]$, and let $n(x)$ be the negation generated by $f(x) : [0,1] \to [0,1]$. The operator $o(x, y)$ satisfies the self-De Morgan property if it satisfies the following equation for all $x, y \in [0,1]$:

$$n(o(x, y) = o(n(x), n(y)).$$

A triangular norm (*t-norm* for short) T is a binary operation on the closed unit interval $[0,1]$ such that $([0,1], T)$ is an abelian semigroup with neutral element 1 which is totally ordered; i.e., for all $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \le x_2$ and $y_1 \le y_2$, we have $T(x_1, y_1) \le T(x_2, y_2)$, where $\le$ is the natural order on $[0,1]$.

A triangular conorm (*t-conorm* for short) S is a binary operation on the closed unit interval $[0,1]$ such that $([0,1], S)$ is an abelian semigroup with neutral element 0 which is totally ordered.