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On the complexity of propositional and relational credal networks ☆

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ABSTRACT

A credal network associates a directed acyclic graph with a collection of sets of probability measures. Usually these probability measures are specified by tables containing probability values. Here we examine the complexity of inference in credal networks when probability measures are specified through formal languages. We focus on logical languages based on propositional logic and on the function-free fragment of first-order logic. We show that sub-Boolean and relational logics lead to interesting complexity results. In short, we explore the relationship between specification language and computational complexity in credal networks.

1. Introduction

A credal network represents a set of probability distributions through a directed acyclic graph and an associated set of "local" credal sets [2,10]. Usually these local credal sets are specified by tables containing probability values, possibly with some additional constraints between them.

In practice, any elicitation strategy must adopt some specification language in which to encode probability assessments and constraints. The simplest specification scheme is to allow inequalities such as $\mathbb{P}(A) \geq 1/2$. Interval-valued assessments such as $\mathbb{P}(A) \in [3/5, 7/10]$ are also common in the literature.

More complex modeling schemes may resort to Boolean operators, relations, and quantifiers. In that case the particular features of the specification language may greatly affect what can and what cannot be represented; also, these features may affect the computational complexity of queries we may wish to pose.

In this paper we study properties of credal networks as parameterized by specification languages, by extending a framework we have recently proposed in the context of Bayesian networks [15]. We look at the interplay between expressivity (of specification languages) and complexity (of inference). We focus on *logical* languages that are fragments of propositional and first-order logic. Additionally, we focus on *strong extensions* as semantics for credal networks. We show that complexity of inferences touches on several interesting complexity classes.

We start with some necessary background in Section 2. We discuss our framework in Section 3, in particular looking at propositional languages. Sections 4 and 5 examine relational languages. Final remarks and possible extensions to this work appear in Section 6.

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2. Background: credal networks and their strong extensions, and some complexity theory

In this paper every possibility space Ω is finite; a random variable is simply a function from Ω into the reals. We use capital letters at the end of the alphabet (X, Y, Z) to denote random variables, and capital letters at the beginning of the alphabet (A, B) to denote events or propositions.

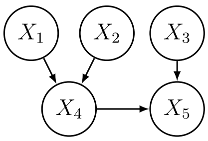
2.1. Credal sets

A set of probability measures is called a *credal set* [36]. When the possibility space Ω is finite, a set of probability distributions over random variables is completely characterized by a set of probability mass functions, each probability mass function corresponding to a distribution. We also use the term *credal set* to refer to sets of probability mass functions.

A set of probability distributions for a variable X is denoted by $\mathbb{K}(X)$. A set of probability mass functions for a variable X is also denoted by $\mathbb{K}(X)$. Given a non-empty credal set $\mathbb{K}$, for any event A we have its *lower* and *upper* probabilities, denoted by $\underline{\mathbb{P}}(A)$ and $\overline{\mathbb{P}}(A)$ respectively: $\underline{\mathbb{P}}(A) = \inf_{\mathbb{P}\in\mathbb{K}} \mathbb{P}(A)$ and $\overline{\mathbb{P}}(A) = \sup_{\mathbb{P}\in\mathbb{K}} \mathbb{P}(A)$. Conditional lower and upper probabilities are defined similarly: For events A and B we define $\underline{\mathbb{P}}(A|B) = \inf_{\mathbb{P}\in\mathbb{K}:\mathbb{P}(B)>0} \mathbb{P}(A|B)$ and $\overline{\mathbb{P}}(A|B) = \sup_{\mathbb{P}\in\mathbb{K}:\mathbb{P}(B)>0} \mathbb{P}(A|B)$ whenever $\overline{\mathbb{P}}(B) > 0$. We leave $\underline{\mathbb{P}}(A|B)$ and $\overline{\mathbb{P}}(A|B)$ undefined when $\overline{\mathbb{P}}(B) = 0$ [60]; in fact, the value of $\underline{\mathbb{P}}(A|B)$ and of $\overline{\mathbb{P}}(A|B)$ is never used in this paper when $\overline{\mathbb{P}}(B) = 0$. An alternative approach would be to consider inference problems under the convention that $\underline{\mathbb{P}}(A|B) = 0$ and $\overline{\mathbb{P}}(A|B) = 1$ whenever $\overline{\mathbb{P}}(B) = 0$ [59]. Another possible approach would be to resort to the theory of full conditional probabilities to obtain $\underline{\mathbb{P}}(A|B)$ and $\overline{\mathbb{P}}(A|B)$ even when $\overline{\mathbb{P}}(B) = 0$ [8]. We leave such possibilities to future work.

2.2. Credal networks

A credal network consists of a directed acyclic graph where each node is a random variable X_i, together with a set of constraints on probability values [2,9,10]. Such a structure is useful as a representation for beliefs, opinions, and statistical summaries that may be available when modeling a particular problem. For instance, suppose we have five variables organized as follows:



Here we have that X_1 and X_2 are *parents* of X_4; likewise, X_3 and X_4 are parents of X_5. The parents of X_i are denoted by $\mathrm{pa}(X_i)$. Intuitively, the graph is understood as indicating that a random variable X_i is directly affected by its parents, and that X_i is independent of its nondescendants nonparents when its parents are fixed (this is called the *Markov condition*).

Even though one is free to impose general constraints on probability values, such as $\mathbb{P}(X_3 = 1, X_2 = 0) \leq 1/2$, usually applications restrict assessments to a few simple forms [2,10]. Typically we have each variable X_i associated with a nonempty *local credal set* $\mathbb{K}_i^{\pi_i}(X_i)$ for each value π_i of the parents of X_i. We assume that each local credal set is closed and convex, as usual in the theory of credal sets [36,59]. Also we focus on local credal sets that are specified by finitely many extreme points,[1] as our specification languages only deal with such objects. In this paper the semantics of such a graph and associated local credal sets is their *strong extension*, defined as the convex hull of the set

$$\left\{ \mathbb{P} : \begin{array}{c} \text{there is } \mathbb{P}(\cdot|\mathrm{pa}(X_i) = \pi_i) \in \mathbb{K}_i^{\pi_i}(X_i) \text{ for each } X_i \text{ and } \pi_i, \text{ such that for every } x_1, \ldots, x_n \\ \text{we have that } \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \prod_i^n \mathbb{P}\left(X_i = x_i | \mathrm{pa}(X_i) = \pi_i'\right) \end{array} \right\}, \tag{1}$$

where π_i' denotes the projection of $\{X_1 = x_1, \ldots, X_n = x_n\}$ on the parents of X_i. In this paper we focus solely on strong extensions, even though there are other ways to interpret the assessments in a credal network [11]. We refer to the set in Expression (1) as the *complete extension* of the credal network.

Note that every extreme point of the strong extension is a product measure consisting of extreme points selected from the local credal sets [21].

Given a credal network (graph and assessments) and its strong extension, we are interested in computing conditional upper probabilities such as $\overline{\mathbb{P}}(X_Q = 1|\mathbf{E})$, for some random variable X_Q and event $\mathbf{E}$.

Suppose we have $\overline{\mathbb{P}}(\mathbf{E}) > 0$ and we want to compute $\overline{\mathbb{P}}(X_Q = 1|\mathbf{E})$. It so happens that we need only look at (the finitely many) extreme points of the strong extension when searching for this latter value, and only at those points that assign positive probability to $\mathbf{E}$. To see this, note that every $\mathbb{P}$ in the strong extension can be written, due to closure and convexity, as a convex combination of extreme points of the strong extension. Hence

[1] An extreme element of a closed convex set is one that cannot be written as a convex combination of other elements in the set.