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On normalization of inconsistency indicators in pairwise comparisons ☆

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In this study, we provide mathematical and practice-driven justification for using $[0, 1]$ normalization of inconsistency indicators in pairwise comparisons. The need for normalization, as well as problems with the lack of normalization, is presented. A new type of paradox of infinity is described.

1. Preliminaries

Consistency in preferences (by pairwise comparisons) can be traced to [1], published by Kendall and Smith in 1939. By 1976, there were four articles listed in [2], with the inconsistency in the title. In [3], the consistency index (CI) was introduced. Many other definitions followed. All of them attempt to answer the most important question: how far have we departed from the consistent state which is well established by the consistency condition illustrated in Fig. 2 in this text. Briefly, every cycle of three interrelated comparisons: A/B, B/C, and A/C must be reducible to two comparisons, since the third comparison is a result of multiplication or division of these two comparisons. For example, given A/B and B/C, A/C should be equal to $(A/B) * (B/C)$. If A/B and A/C are given, B/C is equal to $(A/C)/(A/B)$. If B/C and A/C are given, A/B is equal to $(A/C)/(B/C)$. No other combination exists for a cycle of three comparisons. If all three ratios are given,

☆ Alphabetical order implies equal contribution.

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inconsistency in such cycle may take place and usually, it does. Earlier definitions of inconsistency were based on the total count of inconsistent triads. Evidently, such cardinal inconsistency was imprecise.

This study proposes to regard the inconsistency in pairwise comparisons as a degree or extent of disagreement modelled on the concept of probability measure. Our study also demonstrates a problem shown in [4], where the normalization was not used.

2. Inconsistency indicator as a degree of disagreement in pairwise comparisons

The interval $[0, 1]$ (or discrete values between 0 and 1) is used in many theories and situations including:

- probability,
- multi-valued logic (as proposed by Lukasiewicz),
- fuzzy logic,
- rough set theory,
- theory of evidence.

There are also a number of zero–one laws of which the convergence of conditional expectations, known as "Lévy's zero–one law," can be summarized as "if we are learning gradually all the information that determines the outcome of an event, then we will become gradually certain what the outcome will be," which has application to the consistency-driven pairwise comparisons (recently used in [5], but based on inconsistency indicator introduced in [6]). The important interpretation of "Lévy's zero–one law" is "the gradual data gathering to determine the outcome of an event, then we will become gradually certain what the outcome will be" since it has the direct application to the reduction of inconsistency in pairwise comparisons.

According to [7], P. Lévy proved in 1937 (see [8]), that Kolmogorov's theorem follows from a more general property of conditional probabilities, which says that

$$\lim_{n\to\infty} P\{f(X_1, X_2, \ldots, X_n)|(X_1, X_2, \ldots, X_n)\}, \tag{1}$$

almost certainly equals 0 or 1 (depending on whether $f(X_1, X_2, \ldots X_n)$ is zero or not).

The use of interval $[0, 1]$ for inconsistency indicators is not only coming from the Occam's razor principle. The Section 3 shows that the lack of normalization in [4] leads to problems which this study attempts to correct. In this study, we will follow a business principle "change a problem into opportunity" and use it not only to correct [4] but to reason for an important axiom.

3. The need for normalization of inconsistency indicators

The fallacy of the Definition 3.6 on page 83 in [4]:

Let $X = \langle X, \cdot\rangle$ be a group. A $\mathcal{G}$-distance-based inconsistency indicator map (in abbreviation: a $\mathcal{G}$-inconsistency indicator map) on the group X is a function $T : X^3 \to G$ such that, for all $a, b, c, d, e \in X$, the following conditions are satisfied:
(i) $T(a, b, c) = 1_G \Leftrightarrow ac = b$;
(ii) $T(a, b, c) = T(b, ac, 1)$;
(iii) $T(a, de, c) \le T(a, b, c) \odot T(d, b, e)$.

leads to a contradiction in the theory presented there.

The fallacy in the above definition is evidenced below by using the sequence of triads T_n converging to a consistent matrix when the limit of n, a natural number, is the infinity.

Formally, an inconsistency indicator (ii) is defined for a PC matrix, say M. So, $ii(M)$ should be used. However, a PC matrix:

$$M = \begin{bmatrix} 1 & x & y \\ 1/x & 1 & z \\ 1/y & 1/z & 1 \end{bmatrix}$$

is reduced to the three values: x, y, z which generate the entire matrix. For this reason, we will be using $ii(x, y, z)$ as a notation shortcut (to save on typing and space).

To make our point, all we need to do is demonstrate just one counter-example (for any particular distance). In particular, our goal is achieved when we construct all triads in the sequence T_n this way so that they have constant inconsistency greater than 0 (e.g., one or 2^{64}) and yet such sequence converges. In fact, it is enough to show the convergence for any sequence (e.g., Cauchy sequence). Evidently, T_n is a Cauchy sequence. The well-established consistency condition for PC matrix $M = [m_{ij}]$ is:

$$m_{ik} = m_{ij} * m_{jk} \quad \text{for } i < j < k \le n \tag{2}$$