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Possibility and necessity measures and integral equivalence

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ABSTRACT

Integral equivalence of couples $(\mu, \mathbf{x})$ and $(\mu, \mathbf{y})$, where μ is a possibility (necessity) measure on $[n] = \{1, \ldots, n\}$ and $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ is discussed and studied. We characterize the sets $\mathcal{H}(\mu, \mathbf{x})$ of all $\mathbf{y}$ such that the couples $(\mu, \mathbf{x})$ and $(\mu, \mathbf{y})$ are integral equivalent and we add an illustrative example. Subsequently, a new characterization of possibility (necessity) measures is obtained and the coincidence of universal integrals for possibility (necessity) measures and particular vectors from $[0, 1]^n$ is shown, thus generalizing these results introduced by Dubois and Rico for the Choquet and the Sugeno integrals.

1. Introduction

One distinguished class of utility functions exploited in multiple-criteria decision making is formed by universal integrals on $[0, 1]$ introduced by Klement et al. [6]. They calculate a global evaluation of alternatives characterized by score vectors from $[0, 1]^n$, where n is the number of considered criteria. The most applied integrals are the Choquet integral [1] and the Sugeno integral [9]. Among the other universal integrals on $[0, 1]$, recall the Shilkret integral [8], Weber integral [11] and copula-based integrals [7]. All these discrete integrals are based on a normed monotone measure on the space $[n] = \{1, \ldots, n\}$ named capacity.

Recently, Dubois and Rico [2] have studied the equality of Choquet (Sugeno) integrals of particular couples of score vectors, considering possibility and necessity measures as underlying capacities. Inspired by their results, we extend their problem to all universal integrals on $[0, 1]$, aiming to characterize, for a fixed possibility (necessity) measure and a fixed score vector $\mathbf{x}$, the class of all score vectors with integral values identical to those related to $\mathbf{x}$, independently of the considered universal integral on $[0, 1]$.

The paper is organized as follows. In the next section, we introduce some necessary preliminaries concerning capacities and universal integrals. In Section 3, we study and discuss the above sketched problem considering possibility measures. In Section 4, necessity measures are considered. Finally, some concluding remarks are added.

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2. Preliminaries

For a fixed $n \geq 2$, we denote $[n] = \{1, \ldots, n\}$. For any $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$, we denote by $(\cdot)$ a permutation $(\cdot) : [n] \to [n]$ such that $x_{(1)} \leq \cdots \leq x_{(n)}$, and $x_{(0)} = 0$ by convention. More, we will use notation $A_{(i)} = \{(i), \ldots, (n)\}$. Though the permutation $(\cdot)$ need not be unique (this happens if there are some ties in the sample $(x_1, \ldots, x_n)$), this has no influence on the results presented later. Further, we will use the standard lattice notation $\vee$ for the join (maximum on $[0,1]$) and $\wedge$ for the meet (minimum on $[0,1]$).

Definition 2.1 *([10])*. A monotone set function $\mu : 2^{[n]} \to [0,1]$ is called a capacity whenever it satisfies two boundary conditions $\mu(\emptyset) = 0$ and $\mu([n]) = 1$. A capacity μ is called a possibility (necessity) measure whenever it is maxitive (minitive), i.e., if

$$\mu(A \cup B) = \mu(A) \vee \mu(B) \qquad (\mu(A \cap B) = \mu(A) \wedge \mu(B))$$

for any $A, B \subseteq [n]$. The set of all capacities on $[n]$ will be denoted as $\mathcal{M}_n$.

For any possibility measure Π, the function $\pi : [n] \to [0,1]$, $\pi(i) = \Pi(\{i\})$ is called a possibility distribution (of Π) [12], and for any $A \subseteq [n]$ it holds

$$\Pi(A) = \bigvee_{i \in A} \pi(i),$$

with convention that supremum of the empty set is 0. For any capacity $\mu : 2^{[n]} \to [0,1]$, its dual (conjugate) capacity $\mu^d : 2^{[n]} \to [0,1]$ is given by

$$\mu^d(A) = 1 - \mu([n] \setminus A).$$

Necessity measures are dual to possibility measures, i.e., $N : 2^{[n]} \to [0,1]$ is a necessity measure if and only if its conjugate $N^d = \Pi$ is a possibility measure. Considering the possibility distribution π of Π, it holds

$$N(A) = 1 - \bigvee_{i \notin A} \pi(i).$$

The greatest capacity $\mu^* : 2^{[n]} \to [0,1]$ is given by

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{else} \end{cases},$$

and it is a possibility measure with possibility distribution $\pi^* = 1$.

Its dual μ_*,

$$\mu_*(A) = \begin{cases} 1 & \text{if } A = [n] \\ 0 & \text{else} \end{cases},$$

is a necessity measure, and it is the smallest capacity on $[n]$.

Before introducing the concept of universal integrals on $[0,1]$ we recall the notion of a semicopula.

Definition 2.2 *([3])*. An operation $\otimes : [0,1]^2 \to [0,1]$ is called a semicopula whenever it is increasing in both coordinates and 1 is its neutral element, i.e., $x \otimes 1 = 1 \otimes x = x$ for all $x \in [0,1]$.

The greatest semicopula $\wedge : [0,1]^2 \to [0,1]$ is the standard min operator, $x \wedge y = \min\{x, y\}$. The smallest semicopula is the drastic product T_D,

$$x \, T_D \, y = \begin{cases} x \wedge y & \text{if } x \vee y = 1 \\ 0 & \text{else} \end{cases}.$$

Another distinguished semicopulas are the product T_P, $x \, T_P \, y = x \cdot y$, and the Lukasiewicz t-norm T_L, $x \, T_L \, y = \max\{x + y - 1, 0\}$.

The concept of universal integrals was proposed by Klement et al. [6], and it covers all integrals mentioned in Section 1.

Definition 2.3. Let $\otimes : [0,1]^2 \to [0,1]$ be a fixed semicopula. The mapping $\mathbf{I} : \bigcup_{n \in \mathbb{N}} \mathcal{M}_n \times [0,1]^n \to [0,1]$ is called a ($\otimes$-based) universal integral on $[0,1]$ whenever the next axioms are satisfied: