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Dempster's rule of combination ☆

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ABSTRACT

The theory of belief functions is a generalization of probability theory; a belief function is a set function more general than a probability measure but whose values can still be interpreted as degrees of belief. Dempster's rule of combination is a rule for combining two or more belief functions; when the belief functions combined are based on distinct or "independent" sources of evidence, the rule corresponds intuitively to the pooling of evidence. As a special case, the rule yields a rule of conditioning which generalizes the usual rule for conditioning probability measures. The rule of combination was studied extensively, but only in the case of finite sets of possibilities, in the author's monograph *A Mathematical Theory of Evidence*. The present paper describes the rule for general, possibly infinite, sets of possibilities. We show that the rule preserves the regularity conditions of continuity and condensability, and we investigate the two distinct generalizations of probabilistic independence which the rule suggests.

1. Introduction

A function f defined on a power set $\mathcal{P}(\Omega)$ is called a *belief function* if $f(\emptyset) = 0$, $f(\Omega) = 1$, and f is monotone of order ∞. The author's monograph *A Mathematical Theory of Evidence* [11] studies belief functions in detail under the assumption that Ω is finite. The resulting theory is centered on *Dempster's rule of combination*, a rule for combining two belief functions f_1 and f_2, both defined on the same power set $\mathcal{P}(\Omega)$, to obtain their orthogonal sum $f_1 \oplus f_2$, which is also a belief function on $\mathcal{P}(\Omega)$. This rule is central because of its intuitive interpretation: if f_1 and f_2 express degrees of belief based on entirely distinct bodies of evidence, then the operation of forming $f_1 \oplus f_2$ is interpreted as pooling the two bodies of evidence.

The present paper, which studies Dempster's rule for an arbitrary possibly infinite set Ω, is a sequel to Shafer [13], which studies the representation and extension of belief functions on infinite sets. We freely use the vocabulary, notation and results of that paper.

In preparation for describing Dempster's rule (§6 below), we adduce two special cases: the rule for forming product belief functions (§3) and the rule of conditioning (§5). Both these rules generalize the corresponding rules for probability measures, and both preserve the regularity conditions of continuity and condensability. Like the proof that the product of two countably additive measures is countably additive, the proof that the product of two continuous belief functions is continuous requires a notion of integration—in this case a notion of upper integration which is explained in §2.

In §§4 and 7 we generalize to the case of infinite sets of possibilities the notions of *evidential* and *cognitive independence* of subalgebras with respect to a belief function. Here, as in the finite case (see Chapter 7 of [11]), evidental independence

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requires that both the belief function and its upper probability function obey the usual rule $P(A \cap B) = P(A)P(B)$, while cognitive independence requires only that the upper probability function obey this rule. The notion of evidential independence is intuitively retrospective: two subalgebras are evidentially independent with respect to a belief function if that belief function can be obtained by combining a belief function that bears on only one of them with a belief function that bears on only the other. The weaker notion of cognitive independence, on the other hand, is intuitively prospective: two subalgebras are cognitively independent with respect to a belief function if combination with a new belief function that bears on only one of them does not change the degrees of belief for elements of the other.

It should be noted that the study of monotone and alternating set functions was initiated by Gustave Choquet [3]; and that the theory of belief functions is closely related to Choquet's work; see §2 of Shafer [13]. The theory of belief functions derives, however, from work by A.P. Dempster [4,5], work which was independent of Choquet's work and was directed towards problems in statistical inference. Dempster used the name "lower probabilities" for the quantities here called "degrees of belief," and he introduced his lower and upper probabilities by means of a multivalued mapping from a measure space. He also described his rule of combination in terms of such mappings: see [4]. The axiomatic approach taken here and in [11,13] permits a rigorous study of the rule in connection with the notions of continuity and condensability.

Some of the results in this paper were originally obtained in the author's doctoral dissertation (1973); some were announced in [12]. See also [10].

2. Lower and upper integration

Suppose g is a bounded function on $\mathscr{P}(\Omega)$ such that $g(\emptyset) = 0$ and $g(A) \leqq g(B)$ whenever $A \subset B \subset \Omega$. Given an extended real-valued function φ on Ω, we set

$$G(\varphi) = \int_0^{\infty} g\big(\varphi^{-1}(x, \infty]\big)dx - \int_{-\infty}^{0} \big(g(\Omega) - g\big(\varphi^{-1}(x, \infty]\big)\big)dx \tag{2.1}$$

whenever at least one of the integrals on the right-hand side of this equation is finite. When φ is non-negative, this reduces, of course, to

$$G(\varphi) = \int_0^{\infty} g\big(\varphi^{-1}(x, \infty]\big)dx. \tag{2.2}$$

This functional has been studied extensively, though usually with an emphasis on some topology for Ω; see, for example, [3, p. 265] and [7].

Notice that the set $(x, \infty]$ could be replaced, in (2.1) or (2.2), by the set $[x, \infty]$. For this can change the integrands only at their points of discontinuity, and since they are monotonic, there are only a countable number of these. Notice also that if $A \subset \Omega$ and χ_A is its characteristic function, then $G(\chi_A) = g(A)$; hence the functional G may be thought of as an extension of the set function G. Moreover, if g agrees with a measure μ on some σ-algebra $\mathfrak{A}$ of subsets of Ω and φ is measurable with respect to $\mathfrak{A}$, then $G(\varphi)$ is the integral of φ with respect to μ.

We shall be mainly concerned with the case where φ is non-negative. Accordingly, we denote by Φ^+ the set of all non-negative extended real-valued functions of Ω and note the following facts:

(1) If $\varphi \in \Phi^+$ and $a > 0$ then $G(a\varphi) = aG(\varphi)$.
(2) If $\varphi_1, \varphi_2 \in \Phi^+$ and $\varphi_1 \leqq \varphi_2$, then $G(\varphi_1) \leqq G(\varphi_2)$.
(3) If $g(\bigcap_i A_i) = \lim_{i\to\infty} g(A_i)$ for every decreasing sequence $A_1 \supset A_2 \supset \cdots$ in $\mathscr{P}(\Omega)$, then $G(\lim_{i\to\infty} \varphi_i) = \lim_{i\to\infty} G(\varphi_i)$ for every decreasing sequence $\{\varphi_i\}$ in Φ^+.
(4) If $g(\bigcup_i A_i) = \lim_{i\to\infty} g(A_i)$ for every increasing sequence $A_1 \subset A_2 \subset \cdots$ in $\mathscr{P}(\Omega)$, then $G(\lim_{i\to\infty} \varphi_i) = \lim_{i\to\infty} G(\varphi_i)$ for every increasing sequence $\{\varphi_i\}$ in Φ^+.
(5) If g is monotone of order 2, then

$$G(\varphi_1) + G(\varphi_2) \leqq G(\varphi_1 + \varphi_2)$$

for all $\varphi_1, \varphi_2 \in \Phi^+$.
(6) If g is alternating of order 2, then

$$G(\varphi_1) + G(\varphi_2) \geqq G(\varphi_1 + \varphi_2)$$

for all $\varphi_1, \varphi_2 \in \Phi^+$.

The first four of these facts are easy to verify; (5) and (6) were proven by Choquet [3, pp. 287–288].

A function $\varphi \in \Phi^+$ is called a *simple function* if it can be written in the form