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Solving rank-constrained semidefinite programs in exact arithmetic

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ABSTRACT

We consider the problem of minimizing a linear function over an affine section of the cone of positive semidefinite matrices, with the additional constraint that the feasible matrix has prescribed rank. When the rank constraint is active, this is a non-convex optimization problem, otherwise it is a semidefinite program. Both find numerous applications especially in systems control theory and combinatorial optimization, but even in more general contexts such as polynomial optimization or real algebra. While numerical algorithms exist for solving this problem, such as interior-point or Newton-like algorithms, in this paper we propose an approach based on symbolic computation. We design an exact algorithm for solving rank-constrained semidefinite programs, whose complexity is essentially quadratic on natural degree bounds associated to the given optimization problem: for subfamilies of the problem where the size of the feasible matrix, or the dimension of the affine section, is fixed, the algorithm is polynomial time. The algorithm works under assumptions on the input data: we prove that these assumptions are generically satisfied. We implement it in Maple and discuss practical experiments.

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1. Introduction

1.1. Problem statement

Let $x = (x_1, \ldots, x_n)$ denote a vector of unknowns. We consider the standard semidefinite programming (SDP) problem with additional rank constraints, as follows:

$$(\mathrm{SDP})_r \qquad \begin{aligned} \inf_{x\in\mathbb{R}^n} \ & \ell_c(x) \\ s.t. \ & A(x) \succeq 0 \\ & \operatorname{rank} A(x) \le r \end{aligned} \tag{1}$$

In Problem (1), $\ell_c(x) = c^T x$, $c \in \mathbb{Q}^n$, $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ is a symmetric linear matrix with $A_i \in \mathbb{S}_m(\mathbb{Q})$ (the set of symmetric matrices of size m with entries in $\mathbb{Q}$), and r is an integer, $0 \le r \le m$. The formula $A(x) \succeq 0$ means that $A(x)$ is positive semidefinite (*i.e.*, all its eigenvalues are nonnegative) and is called a linear matrix inequality (LMI). Remark that for $r = m$ this is the standard semidefinite programming problem since the rank constraint is inactive. Moreover, when $c = 0$ (*i.e.*, c is the zero vector), $(\mathrm{SDP})_r$ is a rank-constrained LMI. In the whole paper, we refer to $(\mathrm{SDP})_r$ in Problem (1) as a rank-constrained semidefinite program with parameters (m, n, r). The set

$$\mathscr{S} = \left\{ x \in \mathbb{R}^n : A(x) \succeq 0 \right\},$$

namely the feasible set of $(\mathrm{SDP})_m$, is called a spectrahedron by the convex algebraic geometry community, or equivalently LMI-set. It is a convex basic semialgebraic set. Conversely, for $r < m$, $(\mathrm{SDP})_r$ is no more a convex optimization problem, in general. Indeed, denoted by

$$\mathscr{D}_p = \left\{ x \in \mathbb{C}^n : \operatorname{rank} A(x) \le p \right\}$$

the complex determinantal variety associated to $A(x)$ of maximal rank p, the feasible set of $(\mathrm{SDP})_r$ is exactly $\mathscr{S} \cap \mathscr{D}_r \cap \mathbb{R}^n = \mathscr{S} \cap \mathscr{D}_r$. This is typically non-convex.

The purpose of this paper is to design an exact algorithm for solving problem $(\mathrm{SDP})_r$.

1.2. Contribution

We suppose that the input data is defined over the rational numbers, namely $(c, A_0, A_1, \ldots, A_n) \in \mathbb{Q}^n \times (\mathbb{S}_m(\mathbb{Q}))^{n+1}$. By exact, we mean that, the output of the algorithm is either an empty list, or a finite set S encoded by a rational parametrization as in Rouillier (1999). This is the exact algebraic representation encoded by a vector $(q, q_0, q_1, \ldots, q_n) \subset \mathbb{Q}[t]$ of univariate polynomials, such that q_0, q are coprime and:

$$S = \left\{ \left(\frac{q_1(t)}{q_0(t)}, \ldots, \frac{q_n(t)}{q_0(t)} \right) : q(t) = 0 \right\}. \tag{2}$$

When S is not empty, the degree of q is the algebraic degree of every element in S. When the output is not the empty list, the set S which is returned contains at least one minimizer x^* of $(\mathrm{SDP})_r$. Under general assumptions on input data, which are highlighted and discussed below, the strategy to reach our main goal is twofold:

- we prove that the *semialgebraic* optimization problem $(\mathrm{SDP})_r$ can be reduced to a (*finite*) *sequence* of *algebraic* optimization problems, that is, whose feasible set is real algebraic;
- we design *exact algorithms* for solving the reduced algebraic optimization problems.

Once a rational parametrization $(q, q_0, q_1, \ldots, q_n)$ of S is known, the coordinates of a minimizer can be approximated by intervals of (arbitrary length) of rational numbers, by isolating the real solutions of the univariate equation $q(t) = 0$. The complexity of the real root isolation problem is quadratic in the degree of q and linear in the total bitsize of its coefficients; for more information, *cf.* Pan and Tsigaridas (2015).