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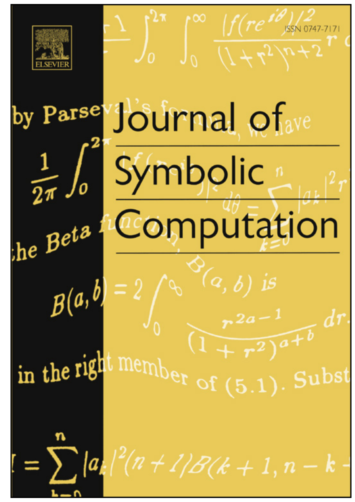

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Harmonic sums and polylogarithms at non-positive multi-indices

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Abstract

Extending Eulerian polynomials and Faulhaber's formula[1], we study several combinatorial aspects of harmonic sums and polylogarithms at non-positive multi-indices as well as their structure. Our techniques are based on the combinatorics of non-commutative generating series in the shuffle Hopf algebras giving a global process to renormalize the divergent polyzetas at non-positive multi-indices.

Keywords: Harmonic sums; Polylogarithms; Bernoulli polynomials; Multi-Eulerian polynomials; Bernoulli numbers, Eulerian numbers.

1. Introduction

The story begins with the celebrated Euler sum [15]

$$\zeta(s) = \sum_{n\geq 1} n^{-s}, \qquad s \in \mathbb{N}, s > 1.$$

Euler gave an explicit formula expressing the following ratio (with $\mathrm{i}^2 = -1$) :

$$\forall j \in \mathbb{N}_+, \quad \frac{\zeta(2j)}{(2\mathrm{i}\pi)^{2j}} = -\frac{1}{2}\frac{b_{2j}}{(2j)!} \quad \in \mathbb{Q}, \tag{1}$$

where $\{b_j\}_{j\in\mathbb{N}}$ are the Bernoulli numbers. Multiplying two such sums, he obtained

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_1 + s_2) + \zeta(s_2, s_1),$$

where the *polyzeta* are given by

$$\zeta(s_1, \ldots, s_r) = \sum_{n_1 > \ldots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r}, \qquad r, s_1, \ldots, s_r \in \mathbb{N}_+, s_1 > 1.$$

[1] First seen and computed up to order 17 by Faulhaber. The modern form and proof are credited to Bernoulli [35].