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Fast computation of the rank profile matrix and the generalized Bruhat decomposition ☆

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The row (resp. column) rank profile of a matrix describes the staircase shape of its row (resp. column) echelon form. We describe a new matrix invariant, the rank profile matrix, summarizing all information on the row and column rank profiles of all the leading sub-matrices. We show that this normal form exists and is unique over a field but also over any principal ideal domain and finite chain ring. We then explore the conditions for a Gaussian elimination algorithm to compute all or part of this invariant, through the corresponding PLUQ decomposition. This enlarges the set of known elimination variants that compute row or column rank profiles. As a consequence a new Crout base case variant significantly improves the practical efficiency of previously known implementations over a finite field. With matrices of very small rank, we also generalize the techniques of Storjohann and Yang to the computation of the rank profile matrix, achieving an $(r^\omega + mn)^{1+o(1)}$ time complexity for an $m \times n$ matrix of rank r, where ω is the exponent of matrix multiplication. Finally, we give connections to the Bruhat decomposition, and several of its variants and generalizations. Consequently, the algorithmic improvements made for the PLUQ factorization, and their implementation, directly apply to these decompositions. In particular, we show how a PLUQ decomposition revealing the rank

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profile matrix also reveals both a row and a column echelon form of the input matrix or of any of its leading sub-matrices, by a simple post-processing made of row and column permutations.
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1. Introduction

Triangular matrix decompositions are widely used in computational linear algebra. Besides solving linear systems of equations, they are also used to compute other objects more specific to exact arithmetic: computing the rank, sampling a vector from the null-space, computing echelon forms and rank profiles.

The *row rank profile* (resp. *column rank profile*) of an $m \times n$ matrix A with rank r, denoted by RowRP(A) (resp. ColRP(A)), is the lexicographically smallest sequence of r indices of linearly independent rows (resp. columns) of A. An $m \times n$ matrix has generic row (resp. column) rank profile if its row (resp. column) rank profile is $(1, .., r)$. Lastly, an $m \times n$ matrix has generic rank profile if its r first leading principal minors are nonzero. Note that if a matrix has generic rank profile, then its row and column rank profiles are generic, but the converse is false: the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ does not have generic rank profile even if its row and column rank profiles are generic. The row support (resp. column support) of a matrix A, denoted by RowSupp(A) (resp. ColSupp(A)), is the subset of indices of its nonzero rows (resp. columns).

We recall that the row echelon form of an $m \times n$ matrix A is an upper triangular matrix $E = TA$, for a nonsingular matrix T, with the zero rows of E at the bottom and the nonzero rows in stair-case shape: $\min\{j : a_{i,j} \neq 0\} < \min\{j : a_{i+1,j} \neq 0\}$. As T is nonsingular, the column rank profile of A is that of E, and therefore corresponds to the column indices of the leading elements in the staircase. Similarly the row rank profile of A is composed of the row indices of the leading elements in the staircase of the column echelon form of A.

Rank profiles and triangular matrix decompositions The rank profiles of a matrix and the triangular matrix decompositions obtained by Gaussian elimination are strongly related. The elimination of matrices with arbitrary rank profiles gives rise to several matrix factorizations and many algorithmic variants. In numerical linear algebra one often uses the PLUQ decomposition, with P and Q permutation matrices, L a lower unit triangular matrix and U an upper triangular matrix. The LSP and LQUP variants of Ibarra et al. (1982) have been introduced to reduce the complexity of rank deficient Gaussian elimination to that of matrix multiplication. Many other algorithmic decompositions exist allowing fraction free computations (Jeffrey, 2010), in-place computations (Dumas et al., 2008; Jeannerod et al., 2013) or sub-cubic rank-sensitive time complexity (Storjohann, 2000; Jeannerod et al., 2013). The reader may refer to Jeannerod et al. (2013) for a detailed comparison between these matrix factorizations, and further details on the CUP (resp. PLE) variants, revealing the row (resp. column) rank profiles. All these algorithms, together with the schoolbook Gaussian elimination algorithm share the property that, for a row rank profile computation, the pivot search processes rows in order, and searches a pivot in all possible column position before declaring the row linearly dependent with the previous ones. As a consequence, blocking is limited to only one dimension (in this case the row dimension) leading to slab algorithms (Klimkowski and van de Geijn, 1995) operating on rectangular blocks of unbalanced dimensions. This reduces the data locality of the algorithm, and therefore penalizes the efficiency of implementations in practice. In parallel, this blocking also puts more constrains on the dependencies between tasks (Dumas et al., 2015a).

Contribution with respect to the state of the art In Dumas et al. (2013) we proposed a first Gaussian elimination algorithm, with a recursive splitting of both row and column dimensions, which simultaneously computes the row and column rank profile while preserving the sub-cubic rank-sensitive

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