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On deflation and multiplicity structure

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ABSTRACT

This paper presents two new constructions related to singular solutions of polynomial systems. The first is a new deflation method for an isolated singular root. This construction uses a single linear differential form defined from the Jacobian matrix of the input, and defines the deflated system by applying this differential form to the original system. The advantages of this new deflation is that it does not introduce new variables and the increase in the number of equations is linear in each iteration instead of the quadratic increase of previous methods. The second construction gives the coefficients of the so-called inverse system or dual basis, which defines the multiplicity structure at the singular root. We present a system of equations in the original variables plus a relatively small number of new variables that completely deflates the root in one step. We show that the isolated simple solutions of this new system correspond to roots of the original system with given multiplicity structure up to a given order. Both constructions are "exact" in that they permit one to treat all conjugate roots simultaneously and can be used in certification procedures for singular roots and their multiplicity structure with respect to an exact rational polynomial system.

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1. Introduction

One issue when using numerical methods for solving polynomial systems is the ill-conditioning and possibly erratic behavior of Newton's method near singular solutions. Regularization (deflation) techniques remove the singular structure to restore local quadratic convergence of Newton's method.

Our motivation for this work is twofold. On one hand, in a recent paper (Akoglu et al., 2014), two of the co-authors of the present paper and their student studied a certification method for approximate roots of exact overdetermined and singular polynomial systems, and wanted to extend the method to certify the multiplicity structure at the root as well. Since all these problems are ill-posed, in Akoglu et al. (2014) a hybrid symbolic-numeric approach was proposed, that included the exact computation of a square polynomial system that had the original root with multiplicity one. In certifying singular roots, this exact square system was obtained from a deflation technique that added subdeterminants of the Jacobian matrix to the system iteratively. However, the multiplicity structure is destroyed by this deflation technique, that is why it remained an open question how to certify the multiplicity structure of singular roots of exact polynomial systems.

Our second motivation was to find a method that simultaneously refines the accuracy of a singular root and the parameters describing the multiplicity structure at the root. In all previous numerical approaches that approximate these parameters, they apply numerical linear algebra to solve a linear system with coefficients depending on the approximation of the coordinates of the singular root. Thus the local convergence rate of the parameters was slowed from the quadratic convergence of Newton's iteration applied to the singular roots. We were interested if the parameters describing the multiplicity structure can be simultaneously approximated with the coordinates of the singular root using Newton's iteration. Techniques that additionally provide information about the multiplicity structure of a singular root can be applied to bifurcation analysis of ODEs and PDEs (see, e.g. Friedman and Hu, 2006; Hao et al., 2012). They can also be helpful in computing the topological degree of a polynomial map (Eisenbud and Levine, 1977) or for analyzing the topology of real algebraic curves (see e.g. Alberti et al., 2008 and Example 6.2 in Mantzaflaris and Mourrain, 2011).

In the present paper, we first give an improved version of the deflation method that can be used in the certification algorithm of Akoglu et al. (2014), reducing the number of added equations at each deflation iteration from quadratic to linear. We prove that applying a single linear differential form to the input system, corresponding to a generic kernel element of the Jacobian matrix, already reduces both the multiplicity and the depth of the singular root. Furthermore, we study how to use this new deflation technique to compute isosingular decompositions introduced in Hauenstein and Wampler (2013).

Secondly, we give a description of the multiplicity structure using a polynomial number of parameters, and express these parameters together with the coordinates of the singular point as the roots of a multivariate polynomial system. We prove that this new polynomial system has a root corresponding to the singular root but now with multiplicity one, and the newly added coordinates describe the multiplicity structure. Thus, this second approach completely deflates the system in one step. The number of equations and variables in the second construction depends polynomially on the number of variables and equations of the input system and the multiplicity of the singular root. Moreover, we also show that the isolated simple solutions of our extended polynomial system correspond to roots of the original system that have prescribed multiplicity structure up to a given order.

Both constructions are exact in the sense that approximations of the coordinates of the singular point are only used to detect numerically non-singular submatrices, and not in the coefficients of the constructed polynomial systems.

This paper is an extended version of the ISSAC'15 conference paper (Hauenstein et al., 2015).

1.1. Related work

The treatment of singular roots is a critical issue for numerical analysis with a large literature on methods that transform the problem into a new one for which Newton-type methods converge quadratically to the root.

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