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Improving root separation bounds

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ABSTRACT

Let f be a polynomial (or polynomial system) with all simple roots. The root separation of f is the minimum of the pair-wise distances between the complex roots. A root separation bound is a lower bound on the root separation. Finding a root separation bound is a fundamental problem, arising in numerous disciplines. We present two new root separation bounds: one univariate bound, and one multivariate bound. The new bounds improve on the old bounds in two ways:

(1) The new bounds are usually *significantly bigger* (hence better) than the previous bounds.
(2) The new bounds *scale correctly*, unlike the previous bounds.

Crucially, the new bounds are *not* harder to compute than the previous bounds.

1. Introduction

In this paper we present improved root separation bounds. A root separation bound is a lower bound on the distances between the roots of a polynomial (or polynomial system). First, we introduce the notion of the root separation by an example.

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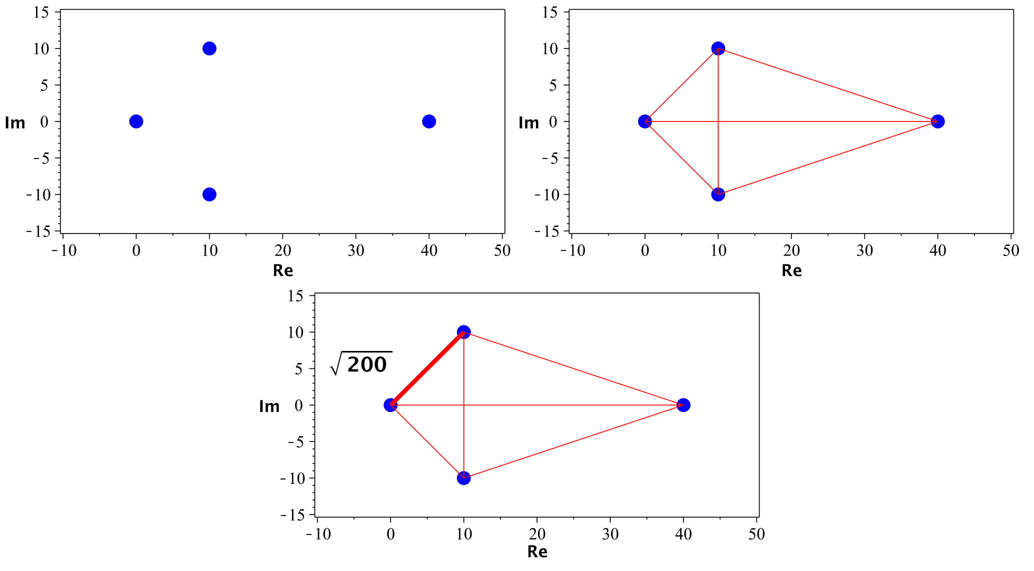

Fig. 1. Roots of $f(x)$ (top left), distances between roots (top right), minimum separation highlighted (bottom center).

Example 1. Let us consider the case of a single polynomial in a single variable. Let $f(x) = x^4 - 60x^3 + 1000x^2 - 8000x$. The roots of $f(x)$ are plotted in Fig. 1. The lengths of the red line segments are the distances between the roots of $f(x)$. The root separation is the smallest of these distances. The root separation of $f(x)$ is $\sqrt{200}$, so any number less than or equal to $\sqrt{200}$ is a root separation bound. □

Root separation bounds are fundamental tools in algorithmic mathematics, with numerous applications (Li et al., 2004; Emiris and Tsigaridas, 2004; Burnikel et al., 2001; Schultz and Moller, 2005; Tsigaridas and Emiris, 2008; Strzebonksi and Tsigaridas, 2011). As a consequence, there has been intensive effort in finding such bounds (Mahler, 1964; Mignotte, 1974, 1995; Rump, 1979; Tsigaridas and Emiris, 2008; Emiris et al., 2010; Collins and Horowitz, 1974; Bugeaud and Mignotte, 2004), resulting in many important bounds. Unfortunately, it is well known that current bounds are very pessimistic. Furthermore, we have found another issue with current bounds. If the roots of a polynomial are doubled, the root separation is obviously doubled. Hence we naturally expect that a root separation bound would double if the roots are doubled. This does not happen: for the polynomial in the above example, the well known Mahler–Mignotte bound (Mahler, 1964; Mignotte, 1974) becomes *smaller* when the roots are doubled. If the roots are tripled, the Mahler–Mignotte bound is *even smaller*. In other words, the Mahler–Mignotte bound does not scale correctly; the bound is not compatible with the geometry of the roots. To the best of the authors' knowledge, the same observation holds for all *efficiently computable* root separation bounds.[1] We elaborate further on this phenomena in the next section.

This discussion leads us to the following challenge: find new root separation bounds such that

(1) the new bounds are (almost always) less pessimistic than previous bounds,
(2) the new bounds scale correctly, and
(3) the new bounds can be computed as efficiently as previous bounds.

[1] There do exist many bounds in the literature which scale correctly but are not efficiently computable. For example, the root separation bound which simply returns the exact root separation scales correctly with the roots. Other examples include the Mahler bound with the Mahler measure in the denominator (Theorem 2 of Mahler, 1964) and several bounds due to Mignotte (1995), all of which depend on the magnitudes of the roots.