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Numerical algorithms for detecting embedded components

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ABSTRACT

We produce algorithms to detect whether a complex affine variety computed and presented numerically by the machinery of numerical algebraic geometry corresponds to an associated component of a polynomial ideal.

1. Introduction

An algorithmic approach to complex algebraic geometry known as *numerical algebraic geometry* (numerical AG, see Sommese et al., 2005; Sommese and Wampler, 2005) provides fast approximate methods to *solve* systems of polynomial equations. In the case when the solution set is a finite set of points *polynomial homotopy continuation* techniques are able to find approximations to all solutions. In the case when the solution set is positive-dimensional, it is a union of irreducible complex affine varieties and *numerical irreducible decomposition* (Sommese et al., 2001) is performed to capture the information about the irreducible pieces with numerical data stored in the so-called *witness sets*. In ideal-theoretic terms, given a generating set of an ideal I in the polynomial ring $R = \mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_N]$, the numerical irreducible decomposition gives a numerical description of the components corresponding to the prime ideals P_i in the decomposition of the radical $\sqrt{I} = P_1 \cap \cdots \cap P_r$.

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The goal of *numerical primary decomposition* (Leykin, 2008) is to find a generic point on every component of the affine scheme $\mathrm{Spec}(R/I)$; in ideal-theoretic terms, find a generic[2] point on the component $\mathbb{V}(P)$ for every associated prime ideal $P \in \mathrm{Ass}(R/I)$. In general a primary decomposition will include *embedded components* not found in an irreducible decomposition, whose corresponding primes strictly contain other associated primes of I.

There are various methods that produce generic points on pieces of the singular locus that may or may not represent embedded components. We refer to such pieces as *suspect components*; if a suspect component does not turn out to be embedded, we call it a *pseudocomponent*.

We see answering the following question algorithmically as one of the first stepping stones to extending numerical AG methods to the full generality of affine schemes.

Problem 1.1. *(Main problem)* For

1. an ideal $I \subset R$ given by a finite generating set,
2. a point $y \in \mathbb{C}^n$, and
3. generic points $y_1, \dots, y_r$ on a collection of components $\mathbb{V}(P_1), \dots, \mathbb{V}(P_r)$, $P_i \in \mathrm{Ass}(R/I)$, that contain y,

decide whether there is a component $\mathbb{V}(P)$, $P \in \mathrm{Ass}(R/I)$, that contains y and is distinct from $\mathbb{V}(P_i)$, for $i = 1, \dots, r$.

We assume that part (1) of the input is *exact*, i.e., the coefficients of the generators of the ideal I are known *exactly*. However, following the framework of numerical AG we assume no access to generators of prime ideals in the parts (2) and (3), nor to exact values for $y, y_1, \dots, y_r$.

We shall describe ideals of a polynomial ring R as well as the ideals of the localization R_y of R at a point $y \in \mathbb{C}^N$ in terms of the *Macaulay dual spaces*.

For convenience, hypothetically, the reader may assume also that the points $y, y_1, \dots, y_r$ in the parts (2) and (3) are exact and the Macaulay dual space algorithm is exact. With this assumption our algorithms become *purely symbolic*. In reality, our approach is hybrid: we state what numerical ingredients are necessary in §5.

The algorithms in this article are implemented in *Macaulay2* (Grayson and Stillman) with parts of code residing in the packages *NumericalHilbert* (Krone, 2014) and *NumericalAlgebraicGeometry* (Leykin, 2011; Leykin). Instructions on steps necessary to reproduce results for the examples are posted at

www.rckr.one/embedded-component-test/.

The beginning of §2 mostly covers basic preliminaries: Macaulay dual spaces and their connection to local polynomial rings, (local) Hilbert function, regularity index, s- and g- corners. Also §2 reviews the operation of taking a colon ideal through the numerical lens and develops the local ideal membership test. The numerical primary decomposition is revisited in §3; this section is not essential, but is used in setting up examples and to provide a better understanding of the general context. The main part of this work, §4, develops algorithms for embedded component testing. One important side result worth highlighting is Theorem 4.15. It concerns associated components of the generic hyperplane section of an affine scheme and makes the dimension reduction possible in our approach. Discussion of numerical ingredients and conclusion is in §5.

[2] Here and throughout the paper we say a "generic point on component" to refer to a point in the complement of a proper Zariski closed subset of the component containing the "degeneracy locus" dictated by the context. One can trust numerical methods mentioned so far to produce random points on components that avoid the degeneracy locus "with probability 1".