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A symbolic decision procedure for relations arising among Taylor coefficients of classical Jacobi theta functions ☆

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ABSTRACT

The class of objects we consider are algebraic relations between the four kinds of classical Jacobi theta functions $\theta_j(z|\tau)$, $j = 1, \dots, 4$, and their derivatives. We present an algorithm to prove such relations automatically where the function argument z is zero, but where the parameter τ in the upper half complex plane is arbitrary.

1. Introduction

The overall objective of this paper is to provide tools for the computer-assisted treatment of identities among Jacobi theta functions. In the first step of development, this amounts to zero-recognition of Taylor coefficients of the respective series expansions of theta functions. To introduce the general idea and application domain of the method presented in this paper, consider the following lemma that has been used in numerous papers like Berndt et al. (1995), Hirschhorn et al. (1993) and Garvan (2010) to prove relations between Jacobi theta series.

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Lemma 1.1. *(Atkin and Swinnerton-Dyer, 1954) Given a non-zero meromorphic function f on $\mathbb{C}\setminus\{0\}$ and $f(wx)\equiv cx^n f(x)$*[1] *for some integer n and non-zero complex constants c and w with $0<|w|<1$, then*

$$\#\,poles(f)=\#\,zeros(f)+n$$

in $|w|<|x|\le 1$.

To do zero recognition of such $f(x)=f(x,q)$, where q is a parameter, the lemma classically is applied as follows: one cleverly chooses sufficiently many zeros $x_1,\dots,x_m$ in the domain $|w|<|x|\le 1$. According to the lemma the number m of such zeros needs to be greater than the number of poles of f minus n, in order to show that f is identically zero. By their clever choice of $x_1,\dots,x_m$, each $f(x_i,q)$ is a modular form when viewed as a function of q. And, zero-recognition of modular forms is algorithmical owing to methods using Sturm bounds or valence formula, e.g., Lemma 4.9 and Proposition 5.13.

Our approach is different and streamlines the idea above by choosing *only one* evaluation point, namely $x_i=1$ for all i, and by verifying that $f^{(j)}(1,q)=0$ for $j\in\{0,\dots,m-1\}$. In this way we prove that there is a zero of multiplicity at least m, which again implies that $f(x)\equiv 0$.

For $j\ge 1$, the Taylor coefficients are not in general modular forms anymore. A crucial point is that, nevertheless, the task of proving relations like $f^{(j)}(1,q)=0$ can again be carried out algorithmically for a large class of problems specified below. The functions that are the building blocks of this class are the Jacobi theta functions $\theta_j(z|\tau)$ $(j=1,\dots,4)$ and their derivatives evaluated at $z=0$. The $\theta_j(z|\tau)$ are defined as follows.

Definition 1.2. (DLMF, 2016) Let $\tau\in\mathbb{H}:=\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}$ and $q=e^{\pi i\tau}$, then

$$\theta_1(z|\tau)=\theta_1(z,q):=2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin((2n+1)z),$$

$$\theta_2(z|\tau)=\theta_2(z,q):=2\sum_{n=0}^{\infty}q^{(n+\frac{1}{2})^2}\cos((2n+1)z),$$

$$\theta_3(z|\tau)=\theta_3(z,q):=1+2\sum_{n=1}^{\infty}q^{n^2}\cos(2nz),$$

$$\theta_4(z|\tau)=\theta_4(z,q):=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz).$$

To exemplify our method of using Lemma 1.1, we consider the following classical example.

Example 1.3. (DLMF, 2016) For $q\in\mathbb{C}$ with $0<|q|<1$, prove

$$\theta_3(0,q)^2\theta_3(z,q)^2-\theta_4(0,q)^2\theta_4(z,q)^2-\theta_2(0,q)^2\theta_2(z,q)^2\equiv 0. \tag{1}$$

Proof. Let $f_j(x):=\theta_j(z,q)$ with $x(z)=e^{2iz}$. Then using the series expansions in Definition 1.2 one can verify directly that $f_j^2(q^2x)=q^{-2}x^{-2}f_j^2(x)$. Define

$$g(x):=\theta_3(0,q)^2 f_3(x)^2-\theta_4(0,q)^2 f_4(x)^2-\theta_2(0,q)^2 f_2(x)^2.$$

Observing that $g(q^2x)=q^{-2}x^{-2}g(x)$, to prove the identity, by Lemma 1.1 it is sufficient to show that $g(x)$ has at least three more zeros than poles in $|q^2|<|x|\le 1$. By Definition 1.2, $g(x)$ has no pole in $\mathbb{C}$. The Taylor expansion of $g(x)$ around $x=1$ is

[1] We use the notation $f_1(z_1,z_2,\dots)\equiv f_2(z_1,z_2,\dots)$ if we want to emphasize that the equality between the functions holds for all possible choices of the arguments z_j.