Journal of Symbolic Computation 81 (2017) 88–96

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

A sharp bound on the number of real intersection points of a sparse plane curve with a line

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ARTICLE INFO

Article history:
Received 8 September 2016
Accepted 3 October 2016
Available online 3 January 2017

MSC:
14T05

Keywords:
Sparse varieties
Fewnomials

ABSTRACT

We prove that the number of real intersection points of a real line with a real plane curve defined by a polynomial with at most t monomials is either infinite or does not exceed $6t - 7$. This improves a result by M. Avendaño. Furthermore, we prove that this bound is sharp for $t = 3$ with the help of Grothendieck's dessins d'enfant.

0. Introduction

The problem of estimating the number of real solutions of a system of polynomial equations is ubiquitous in mathematics and has obvious practical motivations. Fundamental notions like the degree or mixed volume give good estimates for the number of complex solutions of polynomial systems. However, these bounds can overestimate the number of real solutions when the equations have few monomials or a special structure (see Sottile, 2011). In the case of a non-zero single polynomial in one variable, this is a consequence of Descartes' rule of signs which implies that the number of real roots is bounded from above by $2t - 1$, where t is the number of non-zero terms of the polynomial. Generalizations of Descartes' bound for polynomial and more general systems have been obtained by Khovanskii (1991). The resulting bounds for polynomial systems have been improved by Bihan and Sottile (2007), but still very few optimal bounds are known, even in the case of two polynomial equations in two variables. Polynomial systems in two variables where one equation has

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http://dx.doi.org/10.1016/j.jsc.2016.12.003
0747-7171/© 2016 Published by Elsevier Ltd.

three non-zero terms and the other equation has t non-zero terms have been studied by Li et al. (2003). They showed that such a system, allowing real exponents, has at most $2^t - 2$ non-degenerate solutions contained in the positive orthant. This exponential bound has recently been refined into a polynomial one by Koiran et al. (2015). Li et al. (2003) also showed that for $t = 3$ the sharp bound is five. Systems of two trinomial equations with five non-degenerate solutions in the positive orthant are in a sense rare (Dickenstein et al., 2007). Later Avendaño (2009) considered systems of two polynomial equations in two variables, where the first equation has degree one and the other equation has t non-zero terms. He showed that such a system has either an infinite number of real solutions or at most $6t - 4$ real solutions. Here all solutions are counted with multiplicities, with the exception of the solutions on the real coordinate axis which are counted at most once. This reduces to counting the number of real roots of a polynomial $f(x, ax + b)$, where $a, b \in \mathbb{R}$ and $f \in \mathbb{R}[x, y]$ has at most t non-zero terms. The question of optimality was not addressed in Avendaño (2009) and this was the motivation for the present paper. We prove the following result.

Theorem 0.1. *Let $f \in \mathbb{R}[x, y]$ be a polynomial with at most t non-zero terms and let a, b be any real numbers. Assume that the polynomial $g(x) = f(x, ax + b)$ is not identically zero. Then g has at most $6t - 7$ real roots counted with multiplicities except for the possible roots 0 and $-b/a$ that are counted at most once.*

At a first glance this looks like only a slight improvement of the main result of Avendaño (2009). In fact, our bound is optimal at least for $t = 3$.

Theorem 0.2. *The maximal number of real intersection points of a real line with a real plane curve defined by a polynomial with three non-zero terms is eleven.*

Explicitly, the real curve with equation

$$-0.002404\, xy^{18} + 29x^6y^3 + x^3y = 0 \tag{1}$$

intersects the real line $y = x + 1$ in precisely eleven points in $\mathbb{R}^2$.

It is worth noting that to get our bound, as well as the one (Avendaño, 2009), it is sufficient to consider intersection points in the real torus $(\mathbb{R}^*)^2$, so that toric geometry plays a central role here. The strategy to construct the curve (1) is first to deduce from the proof of Theorem 0.1 some necessary conditions on the monomials of the desired equation. Then, the use of real Grothendieck's dessins d'enfant (Bihan, 2007; Brugallé, 2006; Orevkov, 2003) helps to test the feasibility of certain monomials. Ultimately, computer experimentations lead to the precise equation (1).

1. Preliminary results

We present some results of Avendaño (2009) and add other ones. Consider a non-zero univariate polynomial $f(x) = \sum_{i=0}^{d} a_i x^i$ with real coefficients. Denote by $S(f)$ the number of change signs in the ordered sequence $(a_0, \ldots, a_d)$ disregarding the zero terms. Recall that the famous Descartes' rule of signs asserts that the number of (strictly) positive roots of f counted with multiplicities does not exceed $S(f)$.

Lemma 1.1. *(Avendaño, 2009) We have $S((x + 1) f) \leq S(f)$.*

The following two results are straightforward.

Lemma 1.2. *(Avendaño, 2009) If f, g are non-zero polynomials in $\mathbb{R}[x]$ and g has t terms, then $S(f + g) \leq S(f) + 2t$.*

Denote by $\mathcal{N}(h)$ the Newton polytope of a polynomial h and by $\mathcal{N}^\circ(h)$ the interior of $\mathcal{N}(h)$.