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Constructive General Neron Desingularization for one dimensional local rings ☆

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ABSTRACT

An algorithmic proof of General Neron Desingularization is given here for one dimensional local rings and it is implemented in SINGULAR. Also a theorem recalling Greenberg' strong approximation theorem is presented for one dimensional local rings.

1. Introduction

A ring morphism $u : A \to A'$ has *regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ the ring A'/PA' is a regular ring, i.e. its localizations are regular local rings. It has *geometrically regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ and all finite field extensions K of the fraction field of A/P the ring $K \otimes_{A/P} A'/PA'$

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is regular. If for all $P \in \operatorname{Spec} A$ the fraction field of A/P has characteristic 0 then the regular fibers of u are geometrically regular fibers. A flat morphism u is *regular* if its fibers are geometrically regular. If u is regular of finite type then u is called *smooth*. A localization of a smooth algebra is called *essentially smooth*.

In Artin approximation theory (introduced in Artin, 1969) an important result (see Popescu, 2017b) is the following theorem, generalizing the Neron Desingularization (Néron, 1964; Artin, 1969).

Theorem 1 *(General Neron Desingularization, Popescu, 1985, 1986, 1990; André, 1991; Swan, 1998; Spivakovsky, 1999). Let $u : A \to A'$ be a regular morphism of Noetherian rings and B a finite type A-algebra. Then any A-morphism $v : B \to A'$ factors through a smooth A-algebra C, i.e. v is a composite A-morphism $B \to C \to A'$.*

The purpose of this paper is to give an algorithmic proof of the above theorem when A, A' are one dimensional local rings, i.e. Theorem 2. When A, A' are domains such algorithm is given in Popescu and Popescu (in press), the case when A, A' are discrete valuation rings proved by Néron (1964) is given in a different way in Popescu (2017a) with applications in arcs frame. The present algorithm was implemented by the authors in the Computer Algebra system SINGULAR (Decker et al., 2015) and will be as soon as possible found in a development version at https://github.com/Singular/Sources/blob/spielwiese/Singular/LIB/.

The proof of Theorem 2 splits essentially into three steps. We will give here the idea in case A and A' are domains. In step 1 we reduce the problem to the case when $H_{B/A} \cap A \neq 0$, $H_{B/A}$ being the ideal defining the nonsmooth locus of B over A. Let $0 \neq d \in H_{B/A} \cap A$. This means geometrically that $\operatorname{Spec} B_d \to \operatorname{Spec} A_d$ is smooth. In the second step we construct a smooth A-algebra D, $A \subset D \subset A'$ and an A-morphism $v' : B \to D/d^3D$ such that $v \equiv v'$ modulo d^3A'. If A' is the completion $\hat{A}$ of A we can use $D = A$. The third step resolves the singularity. If $B = A[Y]/I$, $Y = (Y_1, \dots, Y_n)$ then we can find $f = (f_1, \dots, f_r)$, $r \leq n$ a system of polynomials from I (given a tuple $(b_1, \dots b_s)$ we denote usually by the corresponding unindexed letter b the vector $(b_1, \dots b_s)$), and an $r \times r$-minor M of the Jacobian matrix $(\partial f_i/\partial Y_j)$ such that $d \equiv MN$ modulo I for some $N \in ((f) : I)$, where (f) denotes the ideal generated by the system f. Then $v'(MN) = ds$ for some $s \in 1 + dD$. Assume that $M = \det(\partial f_i/\partial Y_j)_{1 \leq i,j \leq r}$. Let H be the matrix obtained by adding to $(\partial f/\partial Y)$ the boarder block $(0|\mathrm{Id}_{n-r})$ and let G' be the adjoined matrix of H and $G = NG'$. Consider in $D[Y, T]$, $T = (T_1, \dots, T_n)$, the ideal $J = ((f, s(Y - y') - dG(y')T) : d^2)$, where $y' \in D^n$ is lifting $v'(Y)$. Then C is a suitable localization of the $B \otimes_A D$-algebra $D[Y, T]/(I, J)$ and v extends to C by $v(T) = t = (1/d^2)H(y')(v(Y) - y')$.

Consider the following example. Let $A = \mathbf{Q}[x]_{(x)}$, $A' = \mathbf{C}[[x]]$, $B = A[Y_1, Y_2]/(Y_1^2 + Y_2^2)$, $a \in \mathbf{C}$ a transcendental element over $\mathbf{Q}$, $\bar{u} \in \mathbf{C}[[x]] \setminus \mathbf{C}[x]_{(x)}$ and $u = a + x^6\bar{u}$. Let v be given by $v(Y_1) = xu$, $v(Y_2) = xiu$, where $i = \sqrt{-1}$. In step 1 we change B by $B_1 = A[Y_1, Y_2, Y_3]/I$, $I = (Y_1^2 + Y_2^2, x - 2Y_1Y_3)$ and extend v by $v(Y_3) = 1/(2u)$. We have $4Y_1^2Y_3^2 \in H_{B/A}$ which implies $d = x^2 \in H_{B/A} \cap A$. We define $D = A[a, a^{-1}, i]$ and $v'(Y) = y' = (xa, xia, 1/(2a))$. This is step 2.

To understand step 3 we simplify the example taking $B = A[Y_1, Y_2]/(Y_1Y_2 - x^2)$, $u = 1 + x^6\bar{u}$ and v given by $v(Y_1) = xu$, $v(Y_2) = x/u$. Then $d = x \in H_{B/A}$, $D = A$, $y'_1 = x = y'_2$. We obtain $H = \begin{pmatrix} Y_2 & Y_1 \\ 0 & 1 \end{pmatrix}$, $G = G' = \begin{pmatrix} 1 & -Y_1 \\ 0 & Y_2 \end{pmatrix}$, $N = 1$ and $J = ((Y_1Y_2 - x^2, Y_1 - x - xT_1 + x^2T_2, Y_2 - x - x^2T_2) : x^2)$. We have $J = (xT_1T_2 - x^2T_2^2 + T_1, Y_1 - x - xT_1 + x^2T_2, Y_2 - x - x^2T_2)$ and we obtain that $C \cong (A[T_1, T_2]/(xT_1T_2 - x^2T_2^2 + T_1))_{1+xT_2} \cong (A[T_2])_{1+xT_2}$ is a smooth A-algebra.

When A' is the completion of a Noetherian local ring A of dimension one we show that we may have a linear Artin function as it happens in the Greenberg's case (see Greenberg, 1966 and Popescu and Popescu, in press, Theorem 18). More precisely, the Artin function is given by $c \to (\rho + 1)(e + 1) + c$, where e, ρ depend on A and the polynomial system of equations defining B (see Theorem 14).

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