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A tropical construction of a family of real reducible curves

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ABSTRACT

We give a constructive proof using tropical modifications of the existence of a family of real algebraic plane curves with asymptotically maximal numbers of even ovals.

1. Introduction

Let A be a non-singular real algebraic curve in $\mathbb{CP}^2$. Its real part, denoted by $\mathbb{R}A$, is a disjoint union of embedded circles in $\mathbb{RP}^2$. A component of $\mathbb{R}A$ is called an *oval* if it divides $\mathbb{RP}^2$ in two connected components. If the degree of the curve A is even, then the real part $\mathbb{R}A$ is a disjoint union of ovals. An oval of $\mathbb{R}A$ is called *even* (resp., *odd*) if it is contained inside an even (resp., odd) number of ovals. Denote by p (resp., n) the number of even (resp., odd) ovals of $\mathbb{R}A$.

Petrovski inequalities. For any real algebraic plane curves of degree $2k$, we have

$$-\frac{3}{2}k(k-1) \leq p-n \leq \frac{3}{2}k(k-1)+1.$$

One can deduce upper bounds for p and n from Petrovski inequalities and Harnack theorem (which gives an upper bound for the number of components of a real algebraic curves with respect to its genus):

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$$p \leq \frac{7}{4}k^2 - \frac{9}{4}k + \frac{3}{2},$$

and

$$n \leq \frac{7}{4}k^2 - \frac{9}{4}k + 1.$$

In 1906, V. Ragsdale formulated the following conjecture.

Conjecture *(Ragsdale).* For any real algebraic plane curve of degree $2k$, we have

$$p \leq \frac{3}{2}k(k-1) + 1,$$

and

$$n \leq \frac{3}{2}k(k-1).$$

In 1993, Itenberg (1995) used combinatorial patchworking to construct, for any $k \geq 5$, a real algebraic plane curve of degree $2k$ with

$$p = \frac{3}{2}k(k-1) + 1 + \left\lfloor \frac{(k-3)^2+4}{8} \right\rfloor,$$

and a real algebraic plane curve of degree $2k$ with

$$n = \frac{3}{2}k(k-1) + \left\lfloor \frac{(k-3)^2+4}{8} \right\rfloor.$$

This construction was improved by Haas (1995) then by Itenberg (2001) and finally by Brugallé (2006). Brugallé constructed a family of real algebraic plane curves with

$$\lim_{k \to +\infty} \frac{p}{k^2} = \frac{7}{4},$$

and a family of real algebraic plane curves with

$$\lim_{k \to +\infty} \frac{n}{k^2} = \frac{7}{4}.$$

Brugallé's construction split into two parts. First, he proved the existence of a family of real reducible curves $\mathcal{D}_n \cup \mathcal{C}_n$ in Σ_n, the nth Hirzebruch surface. The curve $\mathcal{D}_n$ has Newton polytope

$$\Delta_n = Conv\left((0,0),(n,0),(0,1)\right),$$

the curve $\mathcal{C}_n$ has Newton polytope

$$\Theta_n = Conv\left((0,0),(n,0),(0,2),(n,1)\right),$$

and the chart of $\mathcal{D}_n \cup \mathcal{C}_n$ is homeomorphic to the one depicted in Fig. 1. We refer to Viro (1984) for the definition of a chart.

Then, he used Viro's patchworking (Viro, 1984) to glue together many copies of the curve obtained by small perturbation of $\mathcal{D}_n \cup \mathcal{C}_n$. Viro's patchworking is a constructive method.

Brugallé's construction of the family $\mathcal{D}_n \cup \mathcal{C}_n$ used *real rational graphs theoretical method*, based on Riemann's existence theorem (Brugallé, 2006; Orevkov, 2003). In particular, this method is not constructive. In this note, we give a constructive method to get such a family using tropical modifications and combinatorial patchworking for complete intersections (Sturmfels, 1994) (see also Theorem 3). In Section 2, we recall basic definitions in tropical geometry. For more details about tropical geometry, we refer to Mikhalkin (2006). In Section 3, we recall the notion of amoebas, the approximation of tropical hypersurfaces by amoebas and the notion of a real phase on a tropical hypersurface. In Section 4, we recall the notion of a tropical modification along a rational function. In Section 5 we give our strategy to construct the family $\mathcal{D}_n \cup \mathcal{C}_n$ and in Sections 6 and 7, we explain the details of our construction.

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