# A certified numerical algorithm for the topology of resultant and discriminant curves 

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#### Abstract

Let $\mathcal{C}$ be a real plane algebraic curve defined by the resultant of two polynomials (resp. by the discriminant of a polynomial). Geometrically such a curve is the projection of the intersection of the surfaces $P(x, y, z)=Q(x, y, z)=0$ (resp. $P(x, y, z)=$ $\frac{\partial P}{\partial z}(x, y, z)=0$ ), and generically its singularities are nodes (resp. nodes and ordinary cusps). State-of-the-art numerical algorithms compute the topology of smooth curves but usually fail to certify the topology of singular ones. The main challenge is to find practical numerical criteria that guarantee the existence and the uniqueness of a singularity inside a given box $B$, while ensuring that $B$ does not contain any closed loop of $\mathcal{C}$. We solve this problem by first providing a square deflation system, based on subresultants, that can be used to certify numerically whether $B$ contains a unique singularity $p$ or not. Then we introduce a numeric adaptive separation criterion based on interval arithmetic to ensure that the topology of $\mathcal{C}$ in $B$ is homeomorphic to the local topology at $p$. Our algorithms are implemented and experiments show their efficiency compared to state-of-the-art symbolic or homotopic methods.


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## 1. Introduction

Given a bivariate polynomial $f$ with rational coefficients, a classical problem is the computation of the topology of the real plane curve $\mathcal{C}=\left\{(x, y) \in \mathbb{R}^{2} \mid f(x, y)=0\right\}$. One may ask for the topology in the whole plane or restricted to some bounding box. In both cases, the topology is output as an embedded piecewise-linear graph that has the same topology as the curve $\mathcal{C}$. For a smooth curve, the graph is hence a collection of topological circles or lines; for a singular curve, the graph must report all the singularities: isolated points and self-intersections.

Symbolic methods based on the cylindrical algebraic decomposition can guarantee the topology of any curve. However, the high complexity of these purely algebraic methods prevents them to be applied in practice on difficult instances. On the other hand, purely numerical methods such as curve tracking with interval arithmetic or subdivision are efficient in practice for smooth curves but typically fail to certify the topology of singular curves. A long-standing challenge is to extend numerical methods to compute efficiently the topology of singular curves.

Computing the topology of a singular curve can be done in three steps.

1. Enclose the singularities in isolating boxes.
2. Compute the local topology in each box, that is i) compute the number of real branches connected to the singularity, ii) ensure that it contains no other branches.
3. Compute the graph connecting the boxes.

Up to homeomorphisms, the local topology at a point of a curve is characterized by the number of real (half-)branches of the curve connected to the point. This number is two for a regular point, and can be any even number at a singular point. We define a witness box of a singular point as a box containing the singular point such that the topology of the curve inside the box is the one of the graph connecting the singularity to the crossings of the curve with the box boundary. The topology of the curve in a witness box is thus completely determined by its number of crossings with the box boundary. In this article, we only focus on the first two steps of the above-mentioned topology algorithm, that is computing witness boxes. The third step can be seen as reporting the topology of a smooth curve in the complement of the witness boxes of the singular points. There already exist certified algorithms for this task using subdivision (Plantinga and Vegter, 2004; Lin and Yap, 2011). Another option is to use certified path tracking (e.g. Martin et al., 2013; Van Der Hoeven, 2011; Beltrán and Leykin, 2013) starting at the crossings of the curve with the boundary of singularity boxes, note that to report the closed loops without singularity, at least one point on such components must be provided.

Contribution and overview. The specificity of the resultant or the discriminant curves computed from generic surfaces is that their singularities are stable, this is a classical result of singularity theory due to Whitney. The key idea of our work is to show that, in this specific case, the over-determined system defining the curve singularities can be transformed into a regular well-constrained system of a transverse intersection of two curves defined by subresultants. This new formulation can be seen as a specific deflation system that does not contain spurious solutions.

Our contribution focuses on the first two steps of the above mentioned topology algorithm for a curve defined by the resultant of two trivariate polynomials $P$ and $Q: f=\operatorname{Resultant}_{z}(P, Q)$, see Fig. 1 for an illustration of the discriminant curve of a torus.

In Section 2, the main results are Theorems 1 and 2 that characterize the singularities of the resultant or discriminant curve in terms of subresultants under generic assumptions. A Semi-algorithm 1 is proposed to check these generic assumptions, i.e. it terminates if and only if the assumptions are satisfied (note that this is the best one can hope for when using a purely numerical method). Based on the characterization of Theorems 1 and 2, Algorithm 2, using subdivision and interval evaluation, isolates the node and cusp singularities with an adaptive certification.

Sections 3 and 4 address the second step on the above-mentioned topology algorithm. Algorithms 3 and 4 in Section 3 distinguish nodes from cusps and compute the number of branches. Then in Section 4, Algorithm 5 refines an isolating box of a singular point such that it becomes a witness box.

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