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Multiple binomial sums

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Multiple binomial sums form a large class of multi-indexed sequences, closed under partial summation, which contains most of the sequences obtained by multiple summation of products of binomial coefficients and also all the sequences with algebraic generating function. We study the representation of the generating functions of binomial sums by integrals of rational functions. The outcome is twofold. Firstly, we show that a univariate sequence is a multiple binomial sum if and only if its generating function is the diagonal of a rational function. Secondly, we propose algorithms that decide the equality of multiple binomial sums and that compute recurrence relations for them. In conjunction with geometric simplifications of the integral representations, this approach behaves well in practice. The process avoids the computation of certificates and the problem of the appearance of spurious singularities that afflicts discrete creative telescoping, both in theory and in practice.

0. Introduction

The computation of definite sums in computer algebra is classically handled by the method of *creative telescoping* initiated in the 1990s by Zeilberger (Zeilberger, 1990, 1991a; Wilf and Zeilberger, 1992). For example, it applies to sums like

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$$\sum_{k=0}^{n} \frac{4^k}{\binom{2k}{k}}, \quad \sum_{k=0}^{n} \left(\sum_{j=0}^{k} \binom{n}{j} \right)^3 \text{ or } \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i+j}{j}^2 \binom{4n-2i-2j}{2n-2i}. \tag{1}$$

In order to compute a sum $\sum_k u(n,k)$ of a bivariate sequence u, this method computes an identity of the form

$$a_p(n)u(n+p,k) + \cdots + a_0(n)u(n,k) = v(n,k+1) - v(n,k).$$

Provided that it is possible to sum both sides over k and that the sequence v vanishes at the end-points of the domain of summation, the left-hand side—called a *telescoper*—gives a recurrence for the sum. The sequence v is then called the *certificate* of the identity.

In the case of multiple sums, this idea leads to searching for a telescoping identity of the form

$$\begin{aligned} a_p(n)u(n+p,k_1,\dots,k_m) + \cdots + a_0(n)u(n,k_1,\dots,k_m) = \\ \big(v_1(n,k_1+1,k_2,\dots,k_m) - v_1(n,k_1,\dots,k_m)\big) + \cdots \\ + \big(v_m(n,k_1,\dots,k_m+1) - v_m(n,k_1,\dots,k_m)\big). \end{aligned} \tag{2}$$

Again, under favorable circumstances the sums of the sequences on the right-hand side telescope, leaving a recurrence for the sum on the left-hand side.

This high-level presentation hides practical difficulties. It is important to check that the sequences on both sides of the identities above are defined over the whole range of summation (Abramov, 2006; Abramov and Petkovšek, 2005). More often than not, singularities do appear. To the best of our knowledge, no algorithm based on creative telescoping manages to work around this difficulty; they all let the user handle it. As a consequence, computing the certificate is not merely a useful by-product of the algorithm, but indeed a necessary part of the computation. Unfortunately, the size of the certificate may be much larger than that of the final recurrence and thus costly in terms of computational complexity.

The computation of multiple integrals of rational functions has some similarities with the computation of discrete sums and the method of creative telescoping applies there too. It may also produce extra singularities in the certificate, but in the differential setting this is not an issue anymore: for the integrals we are interested in, the integration path can always be moved to get around any extra singularity. Moreover, we have showed (Bostan et al., 2013b; Lairez, 2016) that integration of multivariate rational functions over cycles can be achieved efficiently without computing the corresponding certificate and without introducing spurious singularities. In that case, the algorithm computes a linear differential equation for the parameterized integral. It turns out that numerous multiple sums can be cast into problems of rational integration by passing to generating functions. The algorithmic consequences of this observation form the object of the present work.

0.1. Content

In §1, we define a class of multivariate sequences, called *(multiple) binomial sums*, that contains the binomial coefficient sequence and that is closed under pointwise addition, pointwise multiplication, linear change of variables and partial summation. Not every sum that creative telescoping can handle is a binomial sum: for example, among the three sums in Eq. (1), the second one and the third one are binomial sums but the first one is not, since it contains the inverse of a binomial coefficient; moreover, it cannot be rewritten as a binomial sum (see §1.2). Yet many sums coming from combinatorics and number theory are binomial sums. In §2, we explain how to compute *integral representations* of the generating function of a binomial sum in an automated way. The outcome is twofold. Firstly, in §3, we work further on these integral representations to show that the generating functions of *univariate* binomial sums are exactly the diagonals of rational power series. This equivalence characterizes binomial sums in an intrinsic way which dismisses the arbitrariness of the definition. All the theory of diagonals transfers to univariate binomial sums and gives many interesting arithmetic properties. Secondly, in §4, we show how to use integral representations to actually compute with binomial sums