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Monomial right ideals and the Hilbert series of noncommutative modules

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ABSTRACT

In this paper we present a procedure for computing the rational sum of the Hilbert series of a finitely generated monomial right module N over the free associative algebra $\mathbb{K}\langle x_1,\ldots,x_n\rangle$. We show that such procedure terminates, that is, the rational sum exists, when all the cyclic submodules decomposing N are annihilated by monomial right ideals whose monomials define regular formal languages. The method is based on the iterative application of the colon right ideal operation to monomial ideals which are given by an eventual infinite basis. By using automata theory, we prove that the number of these iterations is a minimal one. In fact, we have experimented efficient computations with an implementation of the procedure in Maple which is the first general one for noncommutative Hilbert series.

1. Introduction

It is difficult to list the plenty of objects that are represented as a finite sequence of symbols, that is, by a word or a string. Let us think for example to our own DNA, a message in a code, a word in a natural or formal language, the base expansion of a number, a path in a graph or a transition of states in a machine, etc. From the perspective of the algebraist these elements are just monomials of the free associative algebra $F = \mathbb{K}\langle x_1,\ldots,x_n\rangle$, that is, the algebra of polynomials in the noncommutative variables x_i. In fact, any finitely generated (associative) algebra is isomorphic

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to a quotient F/I where $I \subset F$ is the two-sided ideal of the relations satisfied by the generators of the algebra. Another important generalization is the notion of finitely generated right F-module which corresponds to a quotient F^r/M of the free right F-module $F^r = F \oplus \cdots \oplus F$ with respect to some submodule $M \subset F^r$. It is probably useless to mention the importance of the noncommutative structures, for instance, in physics but let us just refer to the recent book (van Suijlekom, 2015).

Fundamental data about algebras and modules consist in their dimension over the base field $\mathbb{K}$ or in their "growth" in case they are infinite dimensional. By definition, the growth or the affine Hilbert function of $N = F^r/M$ is the map $d \mapsto \dim F^r_{\leq d}/M_{\leq d}$ where $F_{\leq d} \subset F$ is the subspace of the polynomials of degree at most d and $M_{\leq d} = M \cap F^r_{\leq d}$. This sequence is naturally encoded by its generating function which is called the affine Hilbert series of N.

Unlike the commutative case, owing to non-Noetherianity of the free associative algebra one has that this series has generally not a rational sum. Nevertheless, finitely generated free right modules have clearly a rational Hilbert series and the same happens in fact for all finitely presented modules. This is because F is a free ideal ring (Cohn, 2006) which implies that any right submodule $M \subset F^r$ is in fact free. It is then natural to ask if there are other cases when the Hilbert series is rational. One possible approach consists in reducing the problem to the monomial cyclic case that is to right modules $C = F/I$ where $I \subset F$ is a monomial right ideal. This can be done by using a suitable monomial ordering of F^r and by proving a noncommutative analogue of a well-known theorem of Macaulay. We present these results in Section 2.

A short exact sequence for a monomial cyclic right module $C = F/I$ is presented in Section 3 allowing the possibility to reduce the computation of the Hilbert series of C to the one of n cyclic modules defined by the colon right ideals of I with respect to each variable x_i. By iterating this exact sequence one obtains in principle a rational sum for the Hilbert series of C but the main problem is to provide that one has only a finite number of iterations. Under such assumption, in Section 4 we present all details of the corresponding algorithm which is based on the explicit description of a monomial basis for the colon right ideals of a monomial ideal with respect to a monomial of F.

In Section 5 we explain that the set of the colon right ideals of I which are involved in the computation of the Hilbert series of the monomial cyclic right module $C = F/I$ are in fact the states of a minimal deterministic automaton recognizing the formal language given by the monomials of I. By the Kleene's theorem this implies that the proposed procedure has termination if and only if such language is a regular one, that is, all the monomials of I can be obtained from finite subsets by applying a finite number of elementary operations. For the two-sided case, one finds in Mansson (2000), Ufnarovski (1994) the idea that the rationality of the Hilbert series of an algebra $A = F/I$ is provided by the existence of a deterministic finite automaton recognizing the normal monomials modulo I or, equivalently, the monomials of I. In this paper we present a complete and effective direct method for constructing a minimal such automaton. The roots of our approach can be traced back to the formal languages notion of Nerode equivalence that we have enhanced for right and two-sided monomial ideals by means of the tools of algebra.

The proposed ideas result in a feasible algorithm which has been illustrated in Section 6 for an infinitely related monomial algebra that we obtain from an Artin group. In Section 7, by means of an experimental implementation that we have developed with Maple, we present some data and timings for the computation of the Hilbert series of a couple of Hecke algebras. Finally, some conclusions and ideas for further research directions are proposed in Section 8.

2. Hilbert functions and series

Let $\mathbb{K}$ be any field and let $X = \{x_1, \ldots, x_n\}$ be any finite set. We denote by $F = \mathbb{K}\langle X \rangle$ the free associative algebra which is freely generated by X. In other words, the elements of F are noncommutative polynomials in the variables x_i. We define then $W = \mathrm{Mon}(F)$ the set of all monomials of F, that is, the elements of W are words over the alphabet X. We consider F as a graded algebra by means of the standard grading which defines $\deg(w)$ as the length of a word $w \in W$. For any integer $r > 0$, denote by F^r the (finitely generated) free right F-module of rank r. If $\{e_i\}$ is the canonical basis of F^r then the elements of such module are the right F-linear combinations $\sum_i e_i f_i$ $(f_i \in F)$. We endow the module F^r with the standard grading, that is, we put $\deg(e_i w) = \deg(w)$, for any $1 \leq i \leq r$ and

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