Journal of Symbolic Computation 79 (2017) 23–42

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Extreme rays of Hankel spectrahedra for ternary forms

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ARTICLE INFO

Article history:
Received 28 March 2016
Accepted 17 July 2016
Available online 12 August 2016

MSC:
primary 13P25, 14P10, 05E40
secondary 14N99

Keywords:
Sums of squares
Non-negative polynomials
Hankel matrices
Cayley–Bacharach
Spectrahedra
Apolar ideals

ABSTRACT

The cone of sums of squares is one of the central objects in convex algebraic geometry. Its defining linear inequalities correspond to the extreme rays of the dual convex cone. This dual cone is a spectrahedron, which can be explicitly realized as a section of the cone of positive semidefinite matrices with the linear subspace of Hankel (or middle catalecticant) matrices. In this paper we initiate a systematic study of the extreme rays of Hankel spectrahedra for ternary forms. We show that the Zariski closure of the union of extreme rays is the variety of all Hankel matrices of corank at least 4, an irreducible variety of codimension 10 and we determine its degree. We explicitly construct an extreme ray of maximal rank using the Cayley–Bacharach Theorem for plane curves. We apply our results to the study of the algebraic boundary of the cone of sums of squares. Its irreducible components are dual to varieties of Gorenstein ideals with certain Hilbert functions. We determine these Hilbert functions for some cases of small degree. We also observe surprising gaps in the ranks of Hankel matrices of the extreme rays.

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0. Introduction

The following convex cones are fundamental objects in convex algebraic geometry: the cone $P_{n,2d}$ of homogeneous polynomials (forms) of degree $2d$ in $\mathbb{R}[x_1, \ldots, x_n]$ that are nonnegative on $\mathbb{R}^n$, and the cone $\Sigma_{n,2d}$ consisting of sums of squares of degree $2d$. Hilbert showed that only in the following

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http://dx.doi.org/10.1016/j.jsc.2016.08.003
0747-7171/Published by Elsevier Ltd.

three cases every nonnegative form is a sum of squares of forms: bivariate forms, quadratic forms, and ternary forms of degree 4. In all other cases Hilbert showed the existence of nonnegative polynomials that are not sums of squares (Hilbert, 1888).

The dual cones $P_{n,2d}^\vee$ and $\Sigma_{n,2d}^\vee$ consist of all linear functionals nonnegative on the corresponding primal cone. The *extreme rays* of the dual cones provide the defining linear inequalities of the primal cones. Therefore, understanding extreme rays of $\Sigma_{n,2d}^\vee$ is crucial in understanding the boundary of the cone $\Sigma_{n,2d}$, as well as the difference between the cones $P_{n,2d}$ and $\Sigma_{n,2d}$. In the cases where there exist nonnegative polynomials that are not sums of squares, $\Sigma_{n,2d}^\vee$ must contain extreme rays that do not belong to $P_{n,2d}^\vee$. In recent years there has been considerable progress in understanding the extreme rays of $\Sigma_{n,2d}^\vee$ and the *algebraic boundary* of $\Sigma_{n,2d}$, i.e. the Zariski closure of its Euclidean boundary, in the two smallest cases where nonnegative polynomials are not equal to sums of squares: $n = 3$, $2d = 6$ and $n = 4$, $2d = 4$ (Blekherman, 2012; Blekherman et al., 2012). In Blekherman (2012), extreme rays of $\Sigma_{3,6}^\vee$ and $\Sigma_{4,4}^\vee$ were described using the Cayley–Bacharach theorem. In Blekherman et al. (2012), this description led to a quite surprising connection between the algebraic boundaries of $\Sigma_{3,6}$ and $\Sigma_{4,4}$ and moduli spaces of K3 surfaces. In Blekherman (2015), the first author related the study of extreme rays of $\Sigma_{n,2d}^\vee$ to the associated *Gorenstein ideals*.

Taking these results as a point of departure, we begin a systematic study of extreme rays of the cone $\Sigma_{n,2d}^\vee$ for ternary forms, i.e. $n = 3$. We will denote the associated cones simply by Σ_{2d} and $\Sigma_{2d}^\vee$. Our main technical tool will be the Buchsbaum–Eisenbud structure theorem for ternary Gorenstein ideals, and its refined analysis by Diesel (1996). We will see that irreducible components of the algebraic boundary of Σ_{2d} are dual varieties to varieties of Gorenstein ideals with certain Hilbert functions. This gives us a beautiful melding of convex geometry, commutative algebra, and algebraic geometry.

The case of $2d = 6$ was completely described in Blekherman (2012), Blekherman et al. (2012) and therefore we restrict our attention to $2d \geq 8$. Our first main result deals with the Zariski closure of the set of all extreme rays of $\Sigma_{2d}^\vee$ and tells us that extreme rays of $\Sigma_{2d}^\vee$ are plentiful, when compared to extreme rays of $P_{2d}^\vee$.

Theorem *(Theorem 2.15). For any $d \geq 4$, the Zariski closure of the set of extreme rays of $\Sigma_{2d}^\vee$ is the variety of Hankel matrices of corank at least 4. It is irreducible, has codimension 10, and degree $\prod_{\alpha=0}^{3} \binom{N+\alpha}{4-\alpha} / \binom{2\alpha+1}{\alpha}$, where $N = \binom{d+2}{2}$.*

By contrast, the Zariski closure of the extreme rays of $P_{2d}^\vee$ is the $2d$-th Veronese embedding of $\mathbb{P}^2$ and has dimension 2 (Blekherman et al., 2013, Chapter 4). Note that for $\Sigma_6^\vee$, it follows from results of Blekherman (2012), Blekherman et al. (2012) that the Zariski closure of the set of extreme rays is the variety of Hankel matrices of corank at least 3. It has dimension 21, codimension 6 and degree 2640. Existence of extreme rays of co-rank 4 is shown via an intricate explicit construction, which makes heavy use of Cayley–Bacharach theorem for plane curves. The details are given in Section 2.

The extreme rays of the dual cone $\Sigma_{2d}^\vee$ are stratified by the rank of the associated Hankel (middle catalecticant) matrix. This intricate stratification characterizes the algebraic boundary of the sums of squares cone via projective duality theory. We show the following theorem in section 2.

Theorem *(Theorem 2.17). Let X be an irreducible component of the algebraic boundary of Σ_{2d}. Then its dual projective variety X^* is a subvariety of the Zariski closure of the union of extreme rays of $\Sigma_{2d}^\vee$, i.e. the variety of Hankel matrices of corank ≥ 4. Moreover, there is a Hilbert function T such that the quasiprojective variety $\mathrm{Gor}(T)$ of all Gorenstein ideals with Hilbert function T is Zariski dense in X^*.*

We work out the first three nontrivial cases $d = 3, 4, 5$ in Section 3, extending the study of the algebraic boundary of the sums of squares cones for ternary sextics and quaternary quartics in Blekherman et al. (2012). More specifically we show in section 3:

Proposition. *The Hankel spectrahedron $\Sigma_8^\vee$ has extreme rays of rank 1, 10, and 11. We construct extreme rays of rank 10 and 11 such that the Hilbert function of the corresponding Gorenstein ideal is*