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On the auto Igusa-zeta function of an algebraic curve

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ABSTRACT

We study endomorphisms of complete Noetherian local rings in the context of motivic integration. Using the notion of an auto-arc space, we introduce the (reduced) auto-Igusa zeta series at a point, which appears to measure the degree to which a variety is not smooth that point. We conjecture a closed formula in the case of curves with one singular point, and we provide explicit formulas for this series in the case of the cusp and the node. Using the work of Denef and Loeser, one can show that this series will often be rational. These ideas were obtained through extensive calculations in Sage. Thus, we include a Sage script which was used in these calculations. It computes the affine arc spaces $\nabla_{\mathfrak{n}} X$ provided that X is affine, $\mathfrak{n}$ is a fat point, and the ground field is of characteristic zero. Finally, we show that the auto Poincaré series will often be rational as well and connect this to questions concerning new types of motivic integrals.

0. Introduction

We study endomorphisms of complete Noetherian local rings and their connection with motivic integration. More clearly, let $(R, \mathfrak{m})$ be a complete Noetherian local ring with residue field k, then we study the sequence of schemes $\mathcal{A}_n$ which represent the functor $\mathbb{F}\mathfrak{at}_k \to$ **Sets** defined by

$$S \mapsto \mathrm{Mor}_k(\mathrm{Spec}(R/\mathfrak{m}^n) \times_k S, \mathrm{Spec}(R/\mathfrak{m}^n))$$

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where $\mathbb{F}\mathrm{at}_k$ is the full subcategory of separated schemes of finite type over k, denoted here by $\mathbb{S}\mathrm{ch}_k$, whose objects are connected and zero dimensional. Almost always, the schemes $\mathcal{A}_n$ are highly non-reduced for $n \geq 2$. Thus, it is natural to consider their reduction $\mathcal{A}_n^{\mathrm{red}}$. In §2, we describe how we use reduction as a ring homomorphism of two Grothendieck rings – i.e., the Grothendieck ring of the formal site $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_k)$ and the Grothendieck ring of varieties $\mathbf{Gr}(\mathbf{Var}_k)$. In §3, we formally introduce the object $\mathcal{A}_n$, and we name them *auto-arc spaces*. Likewise, in §4, we introduce the auto Igusa-zeta series, which is a generating series for the sequence $\mathcal{A}_n$, with appropriately normalization via negative powers of the Leftschetz motive $\mathbb{L}$. Notably, in §3 and §4, we make conjectures concerning the degree to which auto-arc spaces measure smoothness (resp. étaleness). For example, we conjecture that if $\bar{\zeta}_{X,p}(t) = \mathbb{L}^{-\dim_p(X)} \frac{1}{1-t}$, then X is smooth at p.

In Sections 5, 6, and 7, we investigate this conjecture in the case where X is an algebraic curve. As the spaces $\mathcal{A}_n(X, p)$ are generally quite impossible to compute by hand when n is large, we implement a Sage script the author coded which computes $\mathcal{A}_n(X, p)$. We carry out this computation in §5 and begin to notice some patterns in the case of curves. For example, we notice that if C is the cuspidal cubic (given by $y^2 = x^3$) and O is the origin, then for $n = 4, 5, 6$,

$$\mathcal{A}_n(C, O)^{\mathrm{red}} \cong \nabla_{\mathfrak{l}_{2(n-3)}} C \times_k \mathbb{A}^7_k.$$

In §6, we prove this formula is valid for all $n \geq 4$. Moreover, we prove a similar formula in the case of the node N. This leads us to make a conjecture about the structure of $\mathcal{A}_n(X, p)^{\mathrm{red}}$ when X is an algebraic curve with only one singular point p. Naturally, this leads us in Section 7 to investigate the auto Igusa-zeta series in the case of algebraic curves with only one singular point. We show in the case of smooth curves, the cuspidal cubic, the node, and the nodal cubic, that the auto Igusa-zeta series is intimately connected with the motivic Igusa-zeta series along the linear arc $\mathfrak{l}$ in this case (which is studied in Denef and Loeser, 1998 and further generalized in Schoutens, 2014b). In fact, we explicitly calculate the auto-Igusa zeta function in this case to obtain:

$$(X, p) = (\text{cuspidal cubic, origin}) \implies \bar{\zeta}_{X,p}(t) = \frac{1 - (\mathbb{L}+1)t^3 + \mathbb{L}t^4 + (\mathbb{L}-1)t^5 + 2\mathbb{L}^2 t^6}{(1-\mathbb{L}t^3)(1-t)}$$

$$(X, p) = (\text{node, origin})$$

$$\implies \bar{\zeta}_{X,p}(t) = \frac{1 - (\mathbb{L}^2 + 4\mathbb{L} - 3)t + \mathbb{L}^2(2\mathbb{L}^2 - 1)t^2 - \mathbb{L}^4(3\mathbb{L}^2 - 1)t^3}{(1-\mathbb{L}^2 t)^3}$$

This leads us to make further conjectures about the structure of the auto Igusa-zeta series in the case of algebraic curves which have only one singular point. Whether or not this structural conjecture is true, what is clear is that $\mathcal{A}_n(X, p)^{\mathrm{red}}$ will be a measurable subset of the traditional arc space $\mathcal{L}(W)$ for some algebraic variety W. More work should be carried out in the future to investigate the case of mild singularities of higher dimensional varieties.

In §8, we explore the connection between the potential rationality of the auto Igusa-zeta series and new types of motivic volumes via auto-arc spaces. With a simple adjustment we show that there are such geometric motivic volumes by using the notion of the auto Poincaré series. Here, just as in the previous paragraph, we are making use of the celebrated theorem of J. Denef and F. Loeser, cf. Denef and Loeser (1999), concerning the rationality of motivic Poincaré series and geometric motivic integration. Finally, we use §9 to provide the code used in the computations which occur in §5. Note that the code may be used to calculate any arc space $\nabla_{\mathfrak{n}} X$ provided that X is an affine scheme, $\mathfrak{n}$ is a fat point, and the ground field has characteristic zero. However, the computation speed is destined to be quite slow when the length of $\mathfrak{n}$ is large or when X is complicated.

1. Background

We now quickly give an introduction to Schoutens' theory of schemic Grothendieck rings. Much of what is stated here is taken directly from Schoutens (2014a, 2014b). Let $\mathbb{S}\mathrm{ch}_k$ be the category of separated schemes of finite type over a field k. We say that a functor $\mathcal{X} : \mathbb{S}\mathrm{ch}_k \to \mathbf{Sets}$ is a *sieve* if it is a subfunctor of $X^\circ := \mathrm{Mor}_{\mathbb{S}\mathrm{ch}_k}(-, X)$. We form the Grothendieck pre-topology $\mathbb{F}\mathrm{orm}_k$ on $\mathbb{S}\mathrm{ch}_k$ in the following way.