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The Hurwitz form of a projective variety

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ABSTRACT

The Hurwitz form of a variety is the discriminant that characterizes linear spaces of complementary dimension which intersect the variety in fewer than degree many points. We study computational aspects of the Hurwitz form, relate this to the dual variety and Chow form, and show why reduced degenerations are special on the Hurwitz polytope.

1. Introduction

Many problems in applied algebraic geometry can be expressed as follows. We are given a fixed irreducible variety X in complex projective space $\mathbb{P}^n$ of dimension $d \geq 1$ and degree p. Suppose X is defined over the field $\mathbb{Q}$ of rational numbers. We consider various linear subspaces $L \subset \mathbb{P}^n$ of complementary dimension $n - d$, usually defined over the real numbers in floating point representation. The goal is to compute the intersection $L \cap X$ as accurately as possible. If the subspace L is generic then $L \cap X$ consists of p distinct points with complex coordinates in $\mathbb{P}^n$. How many of the p points are real depends on the specific choice of L.

In this paper we study the discriminant associated with this family of polynomial systems. The precise definition is as follows. Let $\mathrm{Gr}(d, \mathbb{P}^n)$ denote the Grassmannian of codimension d subspaces in $\mathbb{P}^n$, and let $\mathcal{H}_X \subset \mathrm{Gr}(d, \mathbb{P}^n)$ be the subvariety consisting of all subspaces L such that $L \cap X$ does not consist of p reduced points. The *sectional genus* of X, denoted $g = g(X)$, is the arithmetic genus of the curve $X \cap L'$ where $L' \subset \mathbb{P}^n$ is a general subspace of codimension $d - 1$. If X is regular in

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codimension 1 then the curve $X \cap L'$ is smooth (by Bertini's Theorem) and g is its geometric genus. The following result describes our object.

Theorem 1.1. *Let X be an irreducible subvariety of $\mathbb{P}^n$ having degree $p \geq 2$ and sectional genus g. Then $\mathcal{H}_X$ is an irreducible hypersurface in the Grassmannian $\mathrm{Gr}(d, \mathbb{P}^n)$, defined by an irreducible element Hu_X in the coordinate ring of $\mathrm{Gr}(d, \mathbb{P}^n)$. If the singular locus of X has codimension at least 2 then the degree of Hu_X in Plücker coordinates equals $2p + 2g - 2$.*

The polynomial Hu_X defined here is the *Hurwitz form* of X. The name was chosen as a reference to the Riemann–Hurwitz formula, which says that a curve of degree p and genus g has $2p + 2g - 2$ ramification points when mapped onto $\mathbb{P}^1$. We say that $\mathrm{Hdeg}(X) := \deg(\mathrm{Hu}_X) = 2p + 2g - 2$ is the *Hurwitz degree* of X. When X is defined over $\mathbb{Q}$ then so is $\mathcal{H}_X$. Since X is irreducible, its Hurwitz form Hu_X is irreducible. With the geometric definition we have given, Hu_X is unique up to sign, when written in Stiefel coordinates on $\mathrm{Gr}(d, \mathbb{P}^n)$, if we require it to have relatively prime integer coefficients. When written in Plücker coordinates, $\pm\mathrm{Hu}_X$ is unique only modulo the ideal of quadratic Plücker relations.

The Hurwitz form Hu_X belongs to the family of *higher associated hypersurfaces* described by Gel'fand, Kapranov and Zelevinsky in Gel'fand et al. (1994, Section 3.2.E). These hypersurfaces interpolate between the Chow form Ch_X and the X-discriminant. The latter is the equation of the dual variety X^*. In that setting, the Hurwitz form Hu_X is only one step away from the Chow form Ch_X. An important result in Gel'fand et al. (1994, Section 4.3.B) states that these higher associated hypersurfaces are precisely the coisotropic hypersurfaces in $\mathrm{Gr}(d, \mathbb{P}^n)$, so their defining polynomials are governed by the Cayley–Green–Morrison constraints for integrable distributions.

This article is organized as follows. In Section 2 we discuss examples, basic facts, and we derive Theorem 1.1. Section 3 concerns the Hurwitz polytope whose vertices correspond to the initial Plücker monomials of the Hurwitz form. We compare this to the Chow polytope of Kapranov et al. (1992). In Section 4 we define the Hurwitz form of a reduced cycle, and we show that this is compatible with flat families. As an application we resolve problems (4) and (5) in Sanyal et al. (2013, §7).

2. Basics

We begin with examples that illustrate Hurwitz forms in computational algebraic geometry.

Example 2.1 *(Curves).* If X is a curve in $\mathbb{P}^n$, so $d = 1$, then $\mathcal{H}_X = X^*$ is the hypersurface dual to X, and Hu_X is the X-discriminant. For instance, if X is the rational normal curve in $\mathbb{P}^n$ then Hu_X is the discriminant of a polynomial of degree n in one variable. For a curve X in the plane ($n = 2$), the Hurwitz form is the polynomial that defines the dual curve X^*, so the Hurwitz degree $\mathrm{Hdeg}(X)$ is the degree of X^*, which is $p(p-1)$ if X is nonsingular. ◇

Example 2.2 *(Hypersurfaces).* Suppose that X is a hypersurface in $\mathbb{P}^n$, so $d = n - 1$, with defining polynomial $f(x_0, x_1, \ldots, x_n)$. The Grassmannian $\mathrm{Gr}(n-1, \mathbb{P}^n)$ is a manifold of dimension $2n - 2$ in the projective space $\mathbb{P}^{\binom{n+1}{2}-1}$ with dual Plücker coordinates $q_{01}, q_{02}, \ldots, q_{n-1,n}$. We can compute Hu_X by first computing the discriminant of the univariate polynomial function $t \mapsto f(u_0 + tv_0, u_1 + tv_1, \ldots, u_n + tv_n)$, then removing extraneous factors, and finally expressing the result in terms of 2×2-minors via $q_{ij} = u_i v_j - u_j v_i$.

We can make this explicit when $p = 2$. Let M be a symmetric $(n+1) \times (n+1)$-matrix of rank ≥ 2 and X the corresponding quadric hypersurface in $\mathbb{P}^n$. We write $\wedge_2 M$ for the second exterior power of M, and $Q = (q_{01}, q_{02}, \ldots, q_{n-1,n})$ for the row vector of dual Plücker coordinates. With this notation, the Hurwitz form is the following quadratic form in the q_{ij}:

$$\mathrm{Hu}_X \;=\; Q \cdot (\wedge_2 M) \cdot Q^t. \tag{1}$$

For a concrete example let $n = 3$ and consider the quadric surface $X = V(x_0x_3 - x_1x_2)$. Then $\mathrm{Hdeg}(X) = 2$ and the Hurwitz form equals $\mathrm{Hu}_X = q_{03}^2 + q_{12}^2 + 2q_{03}q_{12} - 4q_{02}q_{13}$. When expressed