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Foreword

# What is numerical algebraic geometry?

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ABSTRACT

The foundation of algebraic geometry is the solving of systems of polynomial equations. When the equations to be considered are defined over a subfield of the complex numbers, numerical methods can be used to perform algebraic geometric computations forming the area of numerical algebraic geometry. This article provides a short introduction to numerical algebraic geometry with the subsequent articles in this special issue considering three current research topics: solving structured systems, certifying the results of numerical computations, and performing algebraic computations numerically via Macaulay dual spaces.



## 0. Introduction

Numerical algebraic geometry is concerned with "numerical computations" of objects connected with algebraic sets defined over subfields of the complex numbers. Some examples of such objects include solution sets and irreducible decompositions (Sommese et al., 2001a), monodromy groups (Leykin and Sottile, 2009), and exceptional sets of an algebraic map (Hauenstein and Liddell, 2015; Sommese and Wampler, 2008). The term "numerical" refers to computations which are potentially inexact, e.g., floating-point arithmetic. Another approach to compute similar quantities is to use a symbolic approach, e.g., based on Gröbner basis computations over an algebraic number field or over a prime field of characteristic $p > 0$. There are advantages and disadvantages of each approach, e.g., see Bates et al. (2014) for a comparison of approaches to compute irreducible decompositions. A near-term goal is to design hybrid symbolic-numeric methods that utilize advantages of both approaches.

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Classically, the part of algebraic geometry defined over subfields of the complex numbers is called *transcendental algebraic geometry*. The sets that arise are highly structured and provide many of the basic objects inspiring complex analysis, differential geometry, algebraic topology, and homological algebra.

When floating-point computations are used, at a basic level, one has a finite approximation to all data. A disadvantage of this can be seen with the equation

$$z^2 - 2 = 0. \tag{1}$$

Numerically, a solution may be represented by a numerical approximation such as 1.412 or 1.414213562, neither of which is actually a solution to (1). Due to this, one needs a notion of what it means to numerically compute a solution to system of equations. For a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^N$, suppose that $\xi \in \mathbb{C}^N$ is a nonsingular solution of $f = 0$, i.e., $f(\xi) = 0$ and $Jf(\xi)$ is invertible where $Jf(z)$ is the Jacobian matrix of $f$ evaluated at $z$. One could use the notion of an *approximate solution* associated to $\xi$ based on Newton's method (Blum et al., 1998, Chap. 8) as a definition for "numerically computing" $\xi$. In this case, there is an open neighborhood containing $\xi$, each point of which is an approximate solution of $\xi$, such that Newton's method is a quadratically convergent contraction mapping with $\xi$ being a fixed point of this map. Quadratic convergence yields that $\xi$ can be effectively approximated to any given accuracy starting from an approximate solution.

The two key aspects for defining what it means to "numerically compute" a solution are (1) compute a sufficiently accurate numerical approximation, and (2) have an algorithm that can effectively produce approximations of a true solution to any given accuracy starting from the numerical approximation. Here, "sufficiently accurate" is dependent on the algorithm used to refine the solution, e.g., inside the quadratic convergence basin of Newton's method. In situations related to computing solutions using homotopy continuation, the refining algorithm could be an endgame, e.g., Bates et al. (2011), Huber and Verschelde (1998), Morgan et al. (1991, 1992a, 1992b), where the region of interest is called the endgame operating zone (Sommese and Wampler, 2005, § 3.3.1; Brake et al., 2016). Armed with this information, one is able to recover information about the solution, e.g., deciding reality (Hauenstein and Sottile, 2012) and recovering exact equations (Bates et al., 2013a).

Disadvantages of not working with exact answers can be balanced by an increase in the number of computational methods, e.g., parallelization and typically less memory utilized due to rounding that avoids expression swell. Numerical methods also have the advantage of computing and manipulating individual points on an algebraic set.

Discretization of many systems of nonlinear differential equations lead to large systems of polynomial equations. These discretizations give rise to richly structured partially ordered sets of sparse polynomial systems. Such systems are an important part of transcendental algebraic geometry, to which numerical algebraic geometry naturally applies.

The remaining sections of this introductory article are as follows. Section 1 contains a brief history of numerical algebraic geometry (which is not meant to be exhaustive). Sections 2 and 3 provide a short summary of two main ideas used in this area: path tracking and witness sets, respectively. We conclude in Section 4 with a short summary of the articles in this special issue.

## 1. A brief history

The foundational method which led to the development of numerical algebraic geometry is continuation (often called homotopy continuation). In practice, continuation is simply the ability to track along a solution path (see §2 below).

The first modern uses of continuation, e.g., Hirsch (1963), Kuhn (1968), Scarf (1967), were used to find fixed points with the tracking schemes based on simplicial approximations of the spaces the paths went through. Continuation then evolved into a general method of finding roots of systems of equations, with a major focus in solving differential equations (Eaves and Scarf, 1976; Keller, 1977; Rheinboldt, 1977). A brief overview of this early history, which we followed here, is given in the introduction of Alexander and Yorke (1978). Allgower and Georg's book (Allgower and Georg, 2003)